MATHEMATICS
Module 2
Algebra

TABLE OF CONTENTS

LIST OF FIGURES iii

LIST OF TABLES iv

REFERENCES v

OBJECTIVES vi

ALGEBRAIC LAWS 1

- Algebraic Laws 1
- Summary 3

LINEAR EQUATIONS 4

- Solutions to Algebraic Equations 4
- Algebraic Equations 4
- Types of Algebraic Equations 5
- Linear Equations 6
- Solving Fractional Equations 10
- Ratio and Proportion 13
- Summary 16

QUADRATIC EQUATIONS 17

- Types of Quadratic Equations 17
- Solving Quadratic Equations 17
- Taking Square Root 18
- Factoring Quadratic Equations 21
- The Quadratic Formula 25
- Summary 30

SIMULTANEOUS EQUATIONS 31

- Solving Simultaneous Equations 32
- Summary 41

TABLE OF CONTENTS (Cont)

WORD PROBLEMS 42

Basic Approach to Solving Algebraic Word Problems 42
Steps for Solving Algebraic Word Problems 43
Word Problems Involving Money 50
Problems Involving Motion 54
Solving Word Problems Involving Quadratic Equations 60
Summary 62

LOGARITHMS 63

Calculator Usage, Special Keys 63
Introduction 63
Definition 64
Log Rules 65
Common and Natural Logarithms 68
Anti-Logarithms 69
Natural and Common Log Operations 69
Summary 71

GRAPHING 72

The Cartesian Coordinate System 73
Cartesian Coordinate Graphs 74
Logarithmic Graphs 77
Graphing Equations 81
Nomographs 82
Summary 84

SLOPES 85

Slope 85
Summary 89

INTERPOLATION AND EXTRAPOLATION 90

Definitions 90
Interpolation and Extrapolation 90
Summary 92

LIST OF FIGURES

Figure 1 The Cartesian Coordinate System 73

Figure 2 Cartesian Coordinate Graph of Temperature vs. Time 75

Figure 3 Cartesian Coordinate Graph of Density of Water vs. Temperature 76

Figure 4 Cartesian Coordinate Plot of Radioactive Decay of Strontium 90 78

Figure 5 Semi-log Plot of Radioactive Decay of Strontium 90 79

Figure 6 Log-Log Plot of Frequency vs. Wavelength of Electromagnetic Radiation 80

Figure 7 Plot of x + y = 5 81

Figure 8 Cartesian Coordinate Graph of Quadratic Equation or Function 82

Figure 9 Cartesian Coordinate Graph of Exponential Equation or Function 82

Figure 10 Typical Nomograph 82

Figure 11 Slope 86

LIST OF TABLES

Table 1 Data on the Radioactive Decay of Strontium 90 77

Table 2 Data on Frequency vs. Wavelength of Electromagnetic Radiation 79

REFERENCES

- Dolciani, Mary P., et al., Algebra Structure and Method Book 1, Atlanta: Houghton-Mifflin, 1979.

- Naval Education and Training Command, Mathematics, Volume 1, NAVEDTRA 10069-D1, Washington, D.C.: Naval Education and Training Program Development Center, 1985.

- Science and Fundamental Engineering, Windsor, CT: Combustion Engineering, Inc., 1985.

- Academic Program For Nuclear Power Plant Personnel, Volume 1, Columbia, MD: General Physics Corporation, Library of Congress Card #A 326517, 1982.

TERMINAL OBJECTIVE

1.0 Given a calculator and a list of formulas, **APPLY** the laws of algebra to solve for unknown values.

ENABLING OBJECTIVES

1.1 Given an equation, **DETERMINE** the governing algebraic law from the following:
- a. Commutative law
- b. Associative law
- c. Distributive law

1.2 **SOLVE** for the unknown given a linear equation.

1.3 **APPLY** the quadratic formula to solve for an unknown.

1.4 Given simultaneous equations, **SOLVE** for the unknowns.

1.5 Given a word problem, **WRITE** equations and **SOLVE** for the unknown.

1.6 **STATE** the definition of a logarithm.

1.7 **CALCULATE** the logarithm of a number.

1.8 **STATE** the definition of the following terms:
- a. Ordinate
- b. Abscissa

1.9 Given a table of data, **PLOT** the data points on a cartesian coordinate graph.

1.10 Given a table of data, **PLOT** the data points on a logarithmic coordinate graph.

1.11 Given a table of data, **PLOT** the data points on the appropriate graphing system to obtain the specified curve.

ENABLING OBJECTIVES (Cont)

1.12 **OBTAIN** data from a given graph.

1.13 Given the data, **SOLVE** for the unknown using a nomograph.

1.14 **STATE** the definition of the following terms:
a. Slope
b. Intercept

1.15 Given the equation, **CALCULATE** the slope of a line.

1.16 Given the graph, **DETERMINE** the slope of a line.

1.17 Given a graph, **SOLVE** for the unknown using extrapolation.

1.18 Given a graph, **SOLVE** for the unknown using interpolation.

ALGEBRAIC LAWS

This chapter covers the laws used for solving algebraic equations.

EO 1.1 Given an equation, DETERMINE the governing algebraic law from the following:

a. Commutative law
b. Associative law
c. Distributive law

Most of the work in basic mathematics completed by DOE facility personnel involves real numbers, as mentioned in the last section. As a result, one should be very familiar with the basic laws that govern the use of real numbers. Most of these laws are covered under the general area called Algebra.

Algebraic Laws

Many operations on real numbers are based on the commutative, associative, and distributive laws. The effective use of these laws is important. These laws will be stated in written form as well as algebraic form, where letters or symbols are used to represent an unknown number.

The **commutative laws** indicate that numbers can be added or multiplied in any order.

Commutative Law of Addition: $a + b = b + a$

Commutative Law of Multiplication: $a(b) = b(a)$

The **associative laws** state that in addition or multiplication, numbers can be grouped in any order.

Associative Law of Addition: $a+(b+c) = (a+b)+c$

Associative Law of Multiplication: $a(bc) = (ab)c$

The **distributive laws** involve both addition and multiplication and state the following.

Distributive law: $a(b + c) = ab + ac$

Distributive law: $(a + b)c = ac + bc$

The following list of axioms pertains to the real number system where a, b, and c represent any real numbers. These properties must be true for the algebraic laws to apply.

Closure Properties	1.	$a + b$ is a real number
	2.	ab is a real number
Identity Properties	3.	$a + 0 = a$
	4.	$a(l) = a$
Inverse Properties	5.	For every real number, a, there exists a real number, $-a$, such that $a + (-a) = 0$
	6.	For every real number, $a \neq 0$, there exists a real number, l/a, such that $a\,(1/a) = 1$

An equation is a statement of equality. For example, 4 + 3 = 7. An equation can also be written with one or more unknowns (or variables). The equation x + 7 = 9 is an equality only when the unknown x = 2. The number 2 is called the root or solution of this equation.

The end product of algebra is solving a mathematical equation(s). The operator normally will be involved in the solution of equations that are either linear, quadratic, or simultaneous in nature.

Summary

The important information in this chapter is summarized below.

Algebraic Laws Summary

Commutative Law of Addition	$a + b = b + a$
Commutative Law of Multiplication	$a(b) = b(a)$
Associative Law of Addition	$a+(b+c) = (a+b)+c$
Associative Law of Multiplication	$a(bc) = (ab)c$
Distributive Law	$a(b + c) = ab + ac$

LINEAR EQUATIONS

This chapter covers solving for unknowns using linear equations.

EO 1.2 SOLVE for the unknown given a linear equation.

The rules for addition, subtraction, multiplication, and division described in previous lessons will apply when solving linear equations. Before continuing this course it may be worthwhile to review the basic math laws in Module 1 and the first chapter of this module.

Solutions to Algebraic Equations

The equation is the most important concept in mathematics. Alone, algebraic operations are of little practical value. Only when these operations are coupled with algebraic equations can algebra be applied to solve practical problems.

An equation is a statement of equality between two equal quantities. Most people are familiar with the concept of equality. The idea of equal physical quantities is encountered routinely. An equation is merely the statement of this equality. There are three key ideas in an equation: an equation must involve two expressions, the expressions must be equal, and the equation must indicate that the expressions are equal. Thus, the statement that the sum of three and one equals four is an equation. It involves two expressions, (four and the sum of three and one), the expressions are equal, and the equation states that they are equal.

The equal sign (=) is used to indicate equality in an equation. In its most general form, an algebraic equation consists of two algebraic expressions separated by an equal sign. The equal sign is the key sign in algebra. It is the sign that defines one expression in terms of another. In solving practical problems, it is the sign that defines the unknown quantity in terms of known quantities.

Algebraic Equations

There are two kinds of equations: identities and conditional equations. An identity is an equation that is true for all values of the unknown involved. The identity sign (≡) is used in place of the equal sign to indicate an identity. Thus, $x^2 \equiv (x)(x)$, $3y + 5y \equiv 8y$, and $yx + yz \equiv y(x + z)$ are all identities because they are true for all values of x, y, or z. A conditional equation is one that is true only for some particular value(s) of the literal number(s) involved. A conditional equation is $3x + 5 = 8$, because only the value $x = 1$ satisfies the equation. When the word equation is used by itself, it usually means a conditional equation.

The root(s) of an equation (conditional equation) is any value(s) of the literal number(s) in the equation that makes the equation true. Thus, 1 is the root of the equation $3x + 5 = 8$ because $x = 1$ makes the equation true. To solve an algebraic equation means to find the root(s) of the equation.

The application of algebra is practical because many physical problems can be solved using algebraic equations. For example, pressure is defined as the force that is applied divided by the area over which it is applied. Using the literal numbers P (to represent the pressure), F (to represent the force), and A (to represent the area over which the force is applied), this physical relationship can be written as the algebraic equation $P = \frac{F}{A}$. When the numerical values of the force, F, and the area, A, are known at a particular time, the pressure, P, can be computed by solving this algebraic equation. Although this is a straightforward application of an algebraic equation to the solution of a physical problem, it illustrates the general approach that is used. Almost all physical problems are solved using this approach.

Types of Algebraic Equations

The letters in algebraic equations are referred to as unknowns. Thus, x is the unknown in the equation $3x + 5 = 8$. Algebraic equations can have any number of unknowns. The name unknown arises because letters are substituted for the numerical values that are not known in a problem.

The number of unknowns in a problem determines the number of equations needed to solve for the numerical values of the unknowns. Problems involving one unknown can be solved with one equation, problems involving two unknowns require two independent equations, and so on.

The degree of an equation depends on the power of the unknowns. The degree of an algebraic term is equivalent to the exponent of the unknown. Thus, the term $3x$ is a first degree term; $3x^2$ is a second degree term, and $3x^3$ is a third degree term. The degree of an equation is the same as the highest degree term. Linear or first degree equations contain no terms higher than first degree. Thus, $2x + 3 = 9$ is a linear equation. Quadratic or second degree equations contain up to second degree terms, but no higher. Thus, $x^2 + 3x = 6$, is a quadratic equation. Cubic or third degree equations contain up to third degree terms, but no higher. Thus, $4x^3 + 3x = 12$ is a cubic equation.

The degree of an equation determines the number of roots of the equation. Linear equations have one root, quadratic equations have two roots, and so on. In general, the number of roots of any equation is the same as the degree of the equation.

Exponential equations are those in which the unknown appears in the exponent. For example, $e^{-2.7x} = 290$ is an exponential equation. Exponential equations can be of any degree.

The basic principle used in solving any algebraic equation is: any operation performed on one side of an equation must also be performed on the other side for the equation to remain true. This one principle is used to solve all types of equations.

There are four axioms used in solving equations:

Axiom 1. If the same quantity is added to both sides of an equation, the resulting equation is still true.

Axiom 2. If the same quantity is subtracted from both sides of an equation, the resulting equation is still true.

Axiom 3. If both sides of an equation are multiplied by the same quantity, the resulting equation is still true.

Axiom 4. If both sides of an equation are divided by the same quantity, except 0, the resulting equation is still true.

Axiom 1 is called the addition axiom; Axiom 2, the subtraction axiom; Axiom 3, the multiplication axiom; and Axiom 4, the division axiom. These four axioms can be visualized by the balancing of a scale. If the scale is initially balanced, it will remain balanced if the same weight is added to both sides, if the same weight is removed from both sides, if the weights on both sides are increased by the same factor, or if the weights on both sides are decreased by the same factor.

Linear Equations

These four axioms are used to solve linear equations with three steps:

Step 1. Using the addition and subtraction axioms, Axioms 1 and 2, eliminate all terms with no unknowns from the left-hand side of the equation and eliminate all terms with the unknowns from the right-hand side of the equation.

Step 2. Using the multiplication and division axioms, Axioms 3 and 4, eliminate the coefficient from the unknowns on the left-hand side of the equation.

Step 3. Check the root by substituting it for the unknowns in the original equation.

Example 1:

Solve the equation $3x + 7 = 13$.

Solution:

Step 1. Using Axiom 2, subtract 7 from both sides of the equation.

$$3x + 7 - 7 = 13 - 7$$
$$3x = 6$$

Step 2. Using Axiom 4, divide both sides of the equation by 3.

$$\frac{3x}{3} = \frac{6}{3}$$

$$x = 2$$

Step 3. Check the root.

$$3(2) + 7 = 6 + 7 = 13$$

The root checks.

Example 2:

Solve the equation $2x + 9 = 3(x + 4)$.

Solution:

Step 1. Using Axiom 2, subtract $3x$ and 9 from both sides of the equation.

$$2x + 9 = 3(x + 4)$$
$$2x + 9 - 3x - 9 = 3x + 12 - 3x - 9$$
$$-x = 3$$

Step 2. Using Axiom 4, divide both sides of the equation by -1.

$$\frac{-x}{-1} = \frac{3}{-1}$$
$$x = -3$$

Step 3. Check the root.

$$2(-3) + 9 = -6 + 9 = 3$$
$$3[(-3) + 4] = 3(1) = 3$$

The root checks.

These same steps can be used to solve equations that include several unknowns. The result is an expression for one of the unknowns in terms of the other unknowns. This is particularly important in solving practical problems. Often the known relationship among several physical quantities must be rearranged in order to solve for the unknown quantity. The steps are performed so that the unknown quantity is isolated on the left-hand side of the equation.

Example 1:

Solve the equation $ax - b = c$ for x in terms of a, b, and c.

Solution:

Step 1. Using Axiom 1, add b to both sides of the equation.

$$\begin{aligned} ax - b + b &= c + b \\ ax &= c + b \end{aligned}$$

Step 2. Using Axiom 4, divide both sides of the equation by a.

$$\frac{ax}{a} = \frac{c+b}{a}$$

$$x = \frac{c+b}{a}$$

Step 3. Check the root.

$$a\left(\frac{c+b}{a}\right) - b = c + b - b = c$$

The root checks.

Example 2:

The equation relating the pressure, P, to the force, F, and the area, A, over which the force is applied is $P = \frac{F}{A}$. Solve this equation for F, in terms of P and A.

Solution:

Step 1. Axioms 1 and 2 do not help solve the problem, so go to Step 2.

Step 2. Using Axiom 3, multiply both sides of the equation by A.

$$P(A) = \frac{F}{A}(A)$$

$$F = PA$$

Step 3. Check the root.

$$\frac{PA}{A} = P$$

The root checks.

The addition or subtraction of the same quantity from both sides of an equation may be accomplished by transposing a quantity from one side of the equation to the other. Transposing is a shortened way of applying the addition or subtraction axioms. Any term may be transposed or transferred from one side of an equation to the other if its sign is changed. Thus, in the equation $5x + 4 = 7$, the 4 can be transposed to the other side of the equation by changing its sign. The result is $5x = 7 - 4$ or $5x = 3$. This corresponds to applying the subtraction axiom, Axiom 2, subtracting 4 from both sides of the equation.

Example:

Solve the equation $4x + 3 = 19$ by transposing.

Solution:

Step 1. Transpose the 3 from the left-hand to the right-hand side of the equation by changing its sign.

$$4x = 19 - 3$$

$$4x = 16$$

Step 2. Using Axiom 4, divide both sides of the equation by 4.

$$\frac{4x}{4} = \frac{16}{4}$$

$$x = 4$$

Step 3. Check the root.

$$4(4) + 3 = 16 + 3 = 19$$

The root checks.

Solving Fractional Equations

A fractional equation is an equation containing a fraction. The fraction can be either a common fraction or a decimal fraction. The unknowns can occupy any position in the equation. They may or may not be part of the fraction. If they are part of the fraction, they can be either in the numerator or the denominator. The following are three examples of fractional equations:

$$5x - \frac{1}{2} = 8 \qquad \frac{2x + 6}{3x} = 9 - y \qquad 0.67x + 1.25y = 9$$

Fractional equations are solved using the same axioms and approach used for other algebraic equations. However, the initial step is to remove the equation from fractional form. This is done by determining the lowest common denominator (LCD) for all of the fractions in the equation and then multiplying both sides of the equation by this common denominator. This will clear the equation of fractions.

Example 1:

Solve the fractional equation $\frac{3x + 8}{x} + 5 = 0$.

Solution:

Multiply both sides of the equation by the LCD (x).

$$(x)\left(\frac{3x + 8}{x} + 5\right) = (0)(x)$$

$$3x + 8 + 5x = 0$$

$$8x + 8 = 0$$

Now solve the equation like an ordinary linear equation.

Step 1. Transpose the +8 from the left-hand to the right-hand side of the equation by changing its sign.

$$8x = 0 - 8$$

$$8x = -8$$

Step 2. Using Axiom 4, divide both sides of the equation by 8.

$$\frac{8x}{8} = \frac{-8}{8}$$

$$x = -1$$

Step 3. Check the root.

$$\frac{3(-1) + 8}{-1} + 5 = \frac{-3 + 8}{-1} + 5 = -5 + 5 = 0$$

The root checks.

Example 2:

Solve the fractional equation $\frac{1}{x - 2} + \frac{1}{x + 3} = 0$

Solution:

The LCD is $(x - 2)(x + 3)$; therefore, multiply both sides of the equation by $(x - 2)(x + 3)$.

$$(x - 2)(x + 3)\left(\frac{1}{x - 2} + \frac{1}{x + 3}\right) = (0)(x - 2)(x + 3)$$

$$\frac{(x - 2)(x + 3)}{(x - 2)} + \frac{(x - 2)(x + 3)}{(x + 3)} = 0$$

$$(x + 3) + (x - 2) = 0$$

$$2x + 1 = 0$$

Now solve the equation like an ordinary linear equation.

Step 1. Transpose the +1 from the left-hand to the right-hand side of the equation by changing its sign.

$$2x = 0 - 1$$

$$2x = - 1$$

Step 2. Using Axiom 4, divide both sides of the equation by 2.

$$\frac{2x}{2} = \frac{-1}{2}$$

$$x = -\frac{1}{2}$$

Step 3. Check the root.

$$\frac{1}{-\frac{1}{2}-2}+\frac{1}{-\frac{1}{2}+3}=\frac{1}{-2\frac{1}{2}}+\frac{1}{2\frac{1}{2}}=-\frac{2}{5}+\frac{2}{5}=0$$

The root checks.

Ratio and Proportion

One of the most important applications of fractional equations is ratio and proportion. A ratio is a comparison of two like quantities by division. It is written by separating the quantities by a colon or by writing them as a fraction. To write a ratio, the two quantities compared must be of the same kind. For example, the ratio of \$8 to \$12 is written as \$8:\$12 or $\frac{\$8}{\$12}$. Two unlike quantities cannot be compared by a ratio. For example, 1 inch and 30 minutes cannot form a ratio. However, two different units can be compared by a ratio if they measure the same kind of quantity. For example, 1 minute and 30 seconds can form a ratio, but they must first be converted to the same units. Since 1 minute equals 60 seconds, the ratio of 1 minute to 30 seconds is written 60 seconds:30 seconds, or $\frac{60 \text{ seconds}}{30 \text{ seconds}}$, which equals 2:1 or 2.

A proportion is a statement of equality between two ratios. For example, if a car travels 40 miles in 1 hour and 80 miles in 2 hours, the ratio of the distance traveled is 40 miles:80 miles, or $\frac{40 \text{ miles}}{80 \text{ miles}}$, and the ratio of time is 1 hour:2 hours, or $\frac{1 \text{ hour}}{2 \text{ hours}}$. The proportion relating these two ratios is:

$$40 \text{ miles}:80 \text{ miles} = 1 \text{ hour}:2 \text{ hours}$$

$$\frac{40 \text{ miles}}{80 \text{ miles}} = \frac{1 \text{ hour}}{2 \text{ hours}}$$

A proportion consists of four terms. The first and fourth terms are called the extremes of the proportion; the second and third terms are called the means. If the letters *a*, *b*, *c* and *d* are used to represent the terms in a proportion, it can be written in general form.

$$\frac{a}{b}=\frac{c}{d}$$

Multiplication of both sides of this equation by *bd* results in the following.

$$(bd)\ \frac{a}{b} = \frac{c}{d}\ (bd)$$

$$ad = cb$$

Thus, the product of the extremes of a proportion (*ad*) equals the product of the means (*bc*). For example, in the proportion 40 miles:80 miles = 1 hour:2 hours, the product of the extremes is (40 miles)(2 hours) which equals 80 miles-hours, and the product of the means is (80 miles)(1 hour), which also equals 80 miles-hours.

Ratio and proportion are familiar ideas. Many people use them without realizing it. When a recipe calls for 1½ cups of flour to make a serving for 6 people, and the cook wants to determine how many cups of flour to use to make a serving for 8 people, she uses the concepts of ratios and proportions. When the price of onions is 2 pounds for 49 cents and the cost of 3½ pounds is computed, ratio and proportion are used. Most people know how to solve ratio and proportion problems such as these without knowing the specific steps used.

Ratio and proportion problems are solved by using an unknown such as x for the missing term. The resulting proportion is solved for the value of x by setting the product of the extremes equal to the product of the means.

Example 1:

Solve the following proportion for x.

Solution:

$$5{:}x = 4{:}15$$

The product of the extremes is (5)(15) = 75.
The product of the means is $(x)(4) = 4x$.
Equate these two products and solve the resulting equation.

$$4x = 75$$

$$\frac{4x}{4} = \frac{75}{4}$$

$$x = 18\ \frac{3}{4}$$

Example 2:

If 5 pounds of apples cost 80 cents, how much will 7 pounds cost?

Solution:

Using x for the cost of 7 pounds of apples, the following proportion can be written.

$$\frac{5 \text{ pounds}}{7 \text{ pounds}} = \frac{80 \text{ cents}}{x}$$

The product of the extremes is $(5)(x) = 5x$.
The product of the means is $(7)(80) = 560$.
Equate these two products and solve the resulting equation.

$$5x = 560$$

$$\frac{5x}{5} = \frac{560}{5}$$

$$x = 112$$

The unit of x is cents. Thus, 7 pounds of apples cost 112 cents or \$1.12.

Example 3:

A recipe calls for $1\frac{1}{2}$ cups of flour to make servings for 6 people. How much flour should be used to make servings for 4 people?

Solution:

Using x for the flour required for 4 people, the following proportion can be written.

$$\frac{6 \text{ people}}{4 \text{ people}} = \frac{1\frac{1}{2} \text{ cups}}{x}$$

The product of the extremes is $(6)(x) = 6x$.

The product of the means is $(4)\ 1\frac{1}{2} = 6$.

Equate these two products and solve the resulting equation.

$$6x = 6$$

$$\frac{6x}{6} = \frac{6}{6}$$

$$x = 1$$

The unit of x is cups. Thus, servings for 4 people require 1 cup of flour.

Summary

The important information in this chapter is summarized below.

Linear Equations Summary

There are four axioms used in solving linear equations.

Axiom 1. If the same quantity is added to both sides of an equation, the resulting equation is still true.

Axiom 2. If the same quantity is subtracted from both sides of an equation, the resulting equation is still true.

Axiom 3. If both sides of an equation are multiplied by the same quantity, the resulting equation is still true.

Axiom 4. If both sides of an equation are divided by the same quantity, except 0, the resulting equation is still true.

Axiom 1 is called the addition axiom; Axiom 2, the subtraction axiom; Axiom 3, the multiplication axiom; and Axiom 4, the division axiom.

QUADRATIC EQUATIONS

This chapter covers solving for unknowns using quadratic equations.

EO 1.3 APPLY the quadratic formula to solve for an unknown.

Types of Quadratic Equations

A quadratic equation is an equation containing the second power of an unknown but no higher power. The equation $x^2 - 5x + 6 = 0$ is a quadratic equation. A quadratic equation has two roots, both of which satisfy the equation. The two roots of the quadratic equation $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$. Substituting either of these values for x in the equation makes it true.

The general form of a quadratic equation is the following:

$$ax^2 - bx + c = 0 \tag{2-1}$$

The a represents the numerical coefficient of x^2 , b represents the numerical coefficient of x, and c represents the constant numerical term. One or both of the last two numerical coefficients may be zero. The numerical coefficient a cannot be zero. If b=0, then the quadratic equation is termed a "pure" quadratic equation. If the equation contains both an x and x^2 term, then it is a "complete" quadratic equation. The numerical coefficient c may or may not be zero in a complete quadratic equation. Thus, $x^2 + 5x + 6 = 0$ and $2x^2 - 5x = 0$ are complete quadratic equations.

Solving Quadratic Equations

The four axioms used in solving linear equations are also used in solving quadratic equations. However, there are certain additional rules used when solving quadratic equations. There are three different techniques used for solving quadratic equations: taking the square root, factoring, and the Quadratic Formula. Of these three techniques, only the Quadratic Formula will solve all quadratic equations. The other two techniques can be used only in certain cases. To determine which technique can be used, the equation must be written in general form:

$$ax^2 + bx + c = 0 \tag{2-1}$$

If the equation is a pure quadratic equation, it can be solved by taking the square root. If the numerical constant c is zero, equation 2-1 can be solved by factoring. Certain other equations can also be solved by factoring.

Taking Square Root

A pure quadratic equation can be solved by taking the square root of both sides of the equation. Before taking the square root, the equation must be arranged with the x^2 term isolated on the left-hand side of the equation and its coefficient reduced to 1. There are four steps in solving pure quadratic equations by taking the square root.

Step 1. Using the addition and subtraction axioms, isolate the x^2 term on the left-hand side of the equation.

Step 2. Using the multiplication and division axioms, eliminate the coefficient from the x^2 term.

Step 3. Take the square root of both sides of the equation.

Step 4. Check the roots.

In taking the square root of both sides of the equation, there are two values that satisfy the equation. For example, the square roots of x^2 are $+x$ and $-x$ since $(+x)(+x) = x^2$ and $(-x)(-x) = x^2$. The square roots of 25 are +5 and -5 since (+5)(+5) = 25 and (-5)(-5) = 25. The two square roots are sometimes indicated by the symbol ±. Thus, $\sqrt{25} = \pm 5$. Because of this property of square roots, the two roots of a pure quadratic equation are the same except for their sign.

At this point, it should be mentioned that in some cases the result of solving pure quadratic equations is the square root of a negative number. Square roots of negative numbers are called imaginary numbers and will be discussed later in this section.

Example:

Solve the following quadratic equation by taking the square roots of both sides.

$$3x^2 = 100 - x^2$$

Solution:

Step 1. Using the addition axiom, add x^2 to both sides of the equation.

$$3x^2 + x^2 = 100 - x^2 + x^2$$

$$4x^2 = 100$$

Step 2. Using the division axiom, divide both sides of the equation by 4.

$$\frac{4x^2}{4} = \frac{100}{4}$$

$$x^2 = 25$$

Step 3. Take the square root of both sides of the equation.

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$

$$x = \underline{+}5$$

Thus, the roots are $x = +5$ and $x = -5$.

Step 4. Check the roots.

$$3x^2 = 100 - x^2$$

$$3(\pm 5)^2 = 100 - (\pm 5)^2$$

$$3(25) = 100 - 25$$

$$75 = 75$$

If a pure quadratic equation is written in general form, a general expression can be written for its roots. The general form of a pure quadratic is the following.

$$ax^2 + c = 0 \tag{2-2}$$

Using the subtraction axiom, subtract c from both sides of the equation.

$$ax^2 = -c$$

Using the division axiom, divide both sides of the equation by a.

$$x^2 = -\frac{c}{a}$$

Now take the square roots of both sides of the equation.

$$x = \pm\sqrt{-\frac{c}{a}} \qquad (2\text{-}3)$$

Thus, the roots of a pure quadratic equation written in general form $ax^2 + c = 0$ are $x = +\sqrt{-\frac{c}{a}}$ and $x = -\sqrt{-\frac{c}{a}}$.

Example:

Find the roots of the following pure quadratic equation.

$$4x^2 - 100 = 0$$

Solution:

Using Equation 2-3, substitute the values of c and a and solve for x.

$$x = \pm\sqrt{-\frac{c}{a}}$$

$$x = \pm\sqrt{-\frac{(-100)}{4}}$$

$$x = \pm\sqrt{25}$$

$$x = \pm 5$$

Thus, the roots are $x = 5$ and $x = -5$.

Factoring Quadratic Equations

Certain complete quadratic equations can be solved by factoring. If the left-hand side of the general form of a quadratic equation can be factored, the only way for the factored equation to be true is for one or both of the factors to be zero. For example, the left-hand side of the quadratic equation $x^2 + x - 6 = 0$ can be factored into $(x + 3)(x - 2)$. The only way for the equation $(x + 3)(x - 2) = 0$ to be true is for either $(x + 3)$ or $(x - 2)$ to be zero. Thus, the roots of quadratic equations which can be factored can be found by setting each of the factors equal to zero and solving the resulting linear equations. Thus, the roots of $(x + 3)(x - 2) = 0$ are found by setting $x + 3$ and $x - 2$ equal to zero. The roots are $x = -3$ and $x = 2$.

Factoring estimates can be made on the basis that it is the reverse of multiplication. For example, if we have two expressions $(dx + c)$ and $(cx + g)$ and multiply them, we obtain (using the distribution laws)

$$\begin{aligned}(dx + c)(fx + g) &= (dx)(fx) + (dx)(g) + (c)(fx) + cg = \\ &= dfx^2 + (dg + cf)x + cg.\end{aligned}$$

Thus, a statement $(dx + c)(fx + g) = 0$ can be written

$$df\,x^2 + (dg + cf)x + cg = 0.$$

Now, if one is given an equation $ax^2 + bx + c = 0$, he knows that the symbol a is the product of two numbers (df) and c is also the product of two numbers. For the example $3x^2 - 4x - 4 = 0$, it is a reasonable guess that the numbers multiplying x^2 in the two factors are 3 and 1, although they might be 1.5 and 2. The last -4 (c in the general equation) is the product of two numbers (eg), perhaps -2 and 2 or -1 and 4. These combinations are tried to see which gives the proper value of b $(dg + ef)$, from above.

There are four steps used in solving quadratic equations by factoring.

Step 1. Using the addition and subtraction axioms, arrange the equation in the general quadratic form $ax^2 + bx + c = 0$.

Step 2. Factor the left-hand side of the equation.

Step 3. Set each factor equal to zero and solve the resulting linear equations.

Step 4. Check the roots.

Example:

Solve the following quadratic equation by factoring.

$$2x^2 - 3 = 4x - x^2 + 1$$

Solution:

Step 1. Using the subtraction axiom, subtract $(4x - x^2 + 1)$ from both sides of the equation.

$$2x^2 - 3 - (4x - x^2 + 1) = 4x - x^2 + 1 - (4x - x^2 + 1)$$

$$3x^2 - 4x - 4 = 0$$

Step 2. Factor the resulting equation.

$$3x^2 - 4x - 4 = 0$$

$$(3x + 2)(x - 2) = 0$$

Step 3. Set each factor equal to zero and solve the resulting equations.

$$3x + 2 = 0$$

$$3x = -2$$

$$\frac{3x}{3} = \frac{-2}{3}$$

$$x = -\frac{2}{3}$$

$$x - 2 = 0$$

$$x = 2$$

Thus, the roots are $x = -\frac{2}{3}$ and $x = 2$.

Step 4. Check the roots.

$$2x^2 - 3 = 4x - x^2 + 1$$

$$2\left(-\frac{2}{3}\right)^2 - 3 = 4\left(-\frac{2}{3}\right) - \left(-\frac{2}{3}\right)^2 + 1$$

$$2\left(\frac{4}{9}\right) - 3 = -\frac{8}{3} - \frac{4}{9} + 1$$

$$\frac{8}{9} - \frac{27}{9} = -\frac{24}{9} - \frac{4}{9} + \frac{9}{9}$$

$$-\frac{19}{9} = -\frac{19}{9}$$

$$2x^2 - 3 = 4x - x^2 + 1$$

$$2(2)^2 - 3 = 4(2) - (2)^2 + 1$$

$$2(4) - 3 = 8 - 4 + 1$$

$$8 - 3 = 5$$

$$5 = 5$$

Thus, the roots check.

Quadratic equations in which the numerical constant c is zero can always be solved by factoring. One of the two roots is zero. For example, the quadratic equation $2x^2 + 3x = 0$ can be solved by factoring. The factors are (x) and ($2x + 3$). Thus, the roots are $x = 0$ and $x = -\frac{3}{2}$. If a quadratic equation in which the numerical constant c is zero is written in general form, a general expression can be written for its roots. The general form of a quadratic equation in which the numerical constant c is zero is the following:

$$ax^2 + bx = 0 \qquad (2\text{-}4)$$

The left-hand side of this equation can be factored by removing an x from each term.

$$x(ax + b) = 0 \qquad (2\text{-}5)$$

The roots of this quadratic equation are found by setting the two factors equal to zero and solving the resulting equations.

$$x = 0 \tag{2-6}$$

$$x = -\frac{\mathrm{b}}{\mathrm{a}} \tag{2-7}$$

Thus, the roots of a quadratic equation in which the numerical constant c is zero are $x = 0$ and $x = -\frac{b}{a}$.

Example:

Find the roots of the following quadratic equation.

$$3x^2 + 7x = 0$$

Solution:

Using Equation 2-6, one root is determined.

$$x = 0$$

Using Equation 2-7, substitute the values of *a* and *b* and solve for *x*.

$$x = -\frac{b}{a}$$

$$x = -\frac{7}{3}$$

Thus, the roots are $x = 0$ and $x = -\frac{7}{3}$.

The Quadratic Formula

Many quadratic equations cannot readily be solved by either of the two techniques already described (taking the square roots or factoring). For example, the quadratic equation $x^2 - 6x + 4 = 0$ is not a pure quadratic and, therefore, cannot be solved by taking the square roots. In addition, the left-hand side of the equation cannot readily be factored. The Quadratic Formula is a third technique for solving quadratic equations. It can be used to find the roots of any quadratic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (2\text{-}8)$$

Equation 2-8 is the Quadratic Formula. It states that the two roots of a quadratic equation written in general form, $ax^2 + bx + c = 0$, are equal to $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. The Quadratic Formula should be committed to memory because it is such a useful tool for solving quadratic equations.

There are three steps in solving a quadratic equation using the Quadratic Formula.

Step 1. Write the equation in general form.

Step 2. Substitute the values for *a*, *b*, and *c* into the Quadratic Formula and solve for *x*.

Step 3. Check the roots in the original equation.

Example 1:

Solve the following quadratic equation using the Quadratic Formula.

$4x^2 + 2 = x^2 - 7x$:

Solution:

Step 1. Write the equation in general form.

$$4x^2 + 2 = x^2 - 7x$$

$$3x^2 + 7x + 2 = 0$$

$$a = +3,\ b = +7,\ c = +2$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 - 24}}{6}$$

Step 2.

$$x = \frac{-7 \pm \sqrt{25}}{6}$$

$$x = \frac{-7 \pm 5}{6}$$

$$x = \frac{-7 + 5}{6},\ \frac{-7 - 5}{6}$$

$$x = \frac{-2}{6},\ \frac{-12}{6}$$

$$x = -\frac{1}{3},\ -2$$

Thus, the roots are $x = -\frac{1}{3}$ and $x = -2$.

Step 3. Check the roots.

$$4x^2 + 2 = x^2 - 7x$$

$$4\left(-\frac{1}{3}\right)^2 + 2 = \left(-\frac{1}{3}\right)^2 - 7\left(-\frac{1}{3}\right)$$

$$4\left(\frac{1}{9}\right) + 2 = \frac{1}{9} - \left(-\frac{7}{3}\right)$$

$$\frac{4}{9} + \frac{18}{9} = \frac{1}{9} + \frac{21}{9}$$

$$\frac{22}{9} = \frac{22}{9}$$

and,

$$4x^2 + 2 = x^2 - 7x$$

$$4(-2)^2 + 2 = (-2)^2 - 7(-2)$$

$$4(4) + 2 = 4 - (-14)$$

$$16 + 2 = 4 + 14$$

$$18 = 18$$

Thus, the roots check.

Example 2:

Solve the following quadratic equation using the Quadratic Formula.

$$2x^2 + 4 = 6x + x^2$$

Solution:

Step 1. Write the equation in general form.

$$2x^2 + 4 = 6x + x^2$$

$$x^2 - 6x + 4 = 0$$

$$a = +1,\ b = -6,\ c = +4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 - 16}}{2}$$

$$x = \frac{6 \pm \sqrt{20}}{2}$$

Step 2.

$$x = 3 \pm \frac{1}{2}\sqrt{20}$$

$$x = 3 \pm \frac{1}{2}\sqrt{(4)(5)}$$

$$x = 3 \pm \sqrt{5}$$

$$x = 3 + \sqrt{5},\ 3 - \sqrt{5}$$

$$x = 3 + 2.236,\ 3 - 2.236$$

$$x = 5.236,\ 0.746$$

Step 3. Check the roots.

$$2x^2 + 4 = 6x + x^2$$

$$2(3 + \sqrt{5})^2 + 4 = 6(3 + \sqrt{5}) + (3 + \sqrt{5})^2$$

$$2(9 + 6\sqrt{5} + 5) + 4 = 18 + 6\sqrt{5} + 9 + 6\sqrt{5} + 5$$

$$18 + 12\sqrt{5} + 10 + 4 = 18 + 12\sqrt{5} + 9 + 5$$

$$32 + 12\sqrt{5} = 32 + 12\sqrt{5}$$

and,

$$2x^2 + 4 = 6x + x^2$$

$$2(3 - \sqrt{5})^2 + 4 = 6(3 - \sqrt{5}) + (3 - \sqrt{5})^2$$

$$2(9 - 6\sqrt{5} + 5) + 4 = 18 - 6\sqrt{5} + 9 - 6\sqrt{5} + 5$$

$$18 - 12\sqrt{5} + 10 + 4 = 18 - 12\sqrt{5} + 9 + 5$$

$$32 - 12\sqrt{5} = 32 - 12\sqrt{5}$$

Thus, the roots check.

The Quadratic Formula can be used to find the roots of any quadratic equation. For a pure quadratic equation in which the numerical coefficient b equals zero, the Quadratic Formula (2-8) reduces to the formula given as Equation 2-9.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2-8}$$

For $b = 0$, this reduces to the following.

$$x = \frac{\pm\sqrt{-4ac}}{2a}$$

$$x = \pm\sqrt{\frac{-4ac}{4a^2}}$$

$$x = \pm\sqrt{-\frac{c}{a}} \qquad (2\text{-}9)$$

Summary

The important information in this chapter is summarized below.

Quadratic Equations Summary

There are three methods used when solving quadratic equations:

- Taking the square root
- Factoring the equation
- Using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

SIMULTANEOUS EQUATIONS

This chapter covers solving for two unknowns using simultaneous equations.

EO 1.4 Given simultaneous equations, SOLVE for the unknowns.

Many practical problems that can be solved using algebraic equations involve more than one unknown quantity. These problems require writing and solving several equations, each of which contains one or more of the unknown quantities. The equations that result in such problems are called simultaneous equations because all the equations must be solved simultaneously in order to determine the value of any of the unknowns. The group of equations used to solve such problems is called a system of equations.

The number of equations required to solve any problem usually equals the number of unknown quantities. Thus, if a problem involves only one unknown, it can be solved with a single equation. If a problem involves two unknowns, two equations are required. The equation $x + 3 = 8$ is an equation containing one unknown. It is true for only one value of x: $x = 5$. The equation $x + y = 8$ is an equation containing two unknowns. It is true for an infinite set of xs and ys. For example: $x = 1$, $y = 7$; $x = 2$, $y = 6$; $x = 3$, $y = 5$; and $x = 4$, $y = 4$ are just a few of the possible solutions. For a system of two linear equations each containing the same two unknowns, there is a single pair of numbers, called the solution to the system of equations, that satisfies both equations. The following is a system of two linear equations:

$$2x + y = 9$$
$$x - y = 3$$

The solution to this system of equations is $x = 4$, $y = 1$ because these values of x and y satisfy both equations. Other combinations may satisfy one or the other, but only $x = 4$, $y = 1$ satisfies both.

Systems of equations are solved using the same four axioms used to solve a single algebraic equation. However, there are several important extensions of these axioms that apply to systems of equations. These four axioms deal with adding, subtracting, multiplying, and dividing both sides of an equation by the same quantity. The left-hand side and the right-hand side of any equation are equal. They constitute the same quantity, but are expressed differently. Thus, the left-hand and right-hand sides of one equation can be added to, subtracted from, or used to multiply or divide the left-hand and right-hand sides of another equation, and the resulting equation will still be true. For example, two equations can be added.

$$\begin{array}{rcrcl} & 3x & + & 4y & = 7 \\ + & (x & + & 5y & = 12) \\ \hline & 4x & + & 9y & = 19 \end{array}$$

Adding the second equation to the first corresponds to adding the same quantity to both sides of the first equation. Thus, the resulting equation is still true. Similarly, two equations can be subtracted.

$$\begin{array}{rcrcl} & 4x & - & 3y & = 8 \\ - & (2x & + & 5y & = 11) \\ \hline & 2x & - & 8y & = -3 \end{array}$$

Subtracting the second equation from the first corresponds to subtracting the same quantity from both sides of the first equation. Thus, the resulting equation is still true.

The basic approach used to solve a system of equations is to reduce the system by eliminating the unknowns one at a time until one equation with one unknown results. This equation is solved and its value used to determine the values of the other unknowns, again one at a time. There are three different techniques used to eliminate unknowns in systems of equations: addition or subtraction, substitution, and comparison.

Solving Simultaneous Equations

The simplest system of equations is one involving two linear equations with two unknowns.

$$\begin{aligned} 5x + 6y &= 12 \\ 3x + 5y &= 3 \end{aligned}$$

The approach used to solve systems of two linear equations involving two unknowns is to combine the two equations in such a way that one of the unknowns is eliminated. The resulting equation can be solved for one unknown, and either of the original equations can then be used to solve for the other unknown.

Systems of two equations involving two unknowns can be solved by addition or subtraction using five steps.

Step 1. Multiply or divide one or both equations by some factor or factors that will make the coefficients of one unknown numerically equal in both equations.

Step 2. Eliminate the unknown having equal coefficients by addition or subtraction.

Step 3. Solve the resulting equation for the value of the one remaining unknown.

Step 4. Find the value of the other unknown by substituting the value of the first unknown into one of the original equations.

Step 5. Check the solution by substituting the values of the two unknowns into the other original equation.

Example:

Solve the following system of equations using addition or subtraction.

$$5x + 6y = 12$$

$$3x + 5y = 3$$

Solution:

Step 1. Make the coefficients of y equal in both equations by multiplying the first equation by 5 and the second equation by 6.

$$5(5x + 6y = 12) \text{ yields } 25x + 30y = 60$$

$$6(3x + 5y = 3) \text{ yields } 18x + 30y = 18$$

Step 2. Subtract the second equation from the first.

$$\begin{array}{rrcrcr} & 25x & + & 30y & = & 60 \\ - & (18x & + & 30y & = & 18) \\ \hline & 7x & + & 0 & = & 42 \end{array}$$

Step 3. Solve the resulting equation.

$$7x = 42$$

$$\frac{7x}{7} = \frac{42}{7}$$

$$x = 6$$

Step 4. Substitute $x = 6$ into one of the original equations and solve for y.

$$5x + 6y = 12$$

$$5(6) + 6y = 12$$

$$30 + 6y = 12$$

$$6y = 12 - 30$$

$$6y = -18$$

$$\frac{6y}{6} = \frac{-18}{6}$$

$$y = -3$$

Step 5. Check the solution by substituting $x = 6$ and $y = -3$ into the other original equation.

$$3x + 5y = 3$$

$$3(6) + 5(-3) = 3$$

$$18 - 15 = 3$$

$$3 = 3$$

Thus, the solution checks.

Systems of two equations involving two unknowns can also be solved by substitution.

Step 1. Solve one equation for one unknown in terms of the other.

Step 2. Substitute this value into the other equation.

Step 3. Solve the resulting equation for the value of the one remaining unknown.

Step 4. Find the value of the other unknown by substituting the value of the first unknown into one of the original equations.

Step 5. Check the solution by substituting the values of the two unknowns into the other original equation.

Example:

Solve the following system of equations using substitution.

$$5x + 6y = 12$$
$$3x + 5y = 3$$

Solution:

Step 1. Solve the first equation for x.

$$5x + 6y = 12$$

$$5x = 12 - 6y$$

$$\frac{5x}{5} = \frac{12 - 6y}{5}$$

$$x = \frac{12}{5} - \frac{6y}{5}$$

Step 2. Substitute this value of x into the second equation.

$$3x + 5y = 3$$

$$3\left(\frac{12}{5} - \frac{6y}{5}\right) + 5y = 3$$

Step 3. Solve the resulting equation.

$$3\left(\frac{12}{5} - \frac{6y}{5}\right) + 5y = 3$$

$$\frac{36}{5} - \frac{18y}{5} + 5y = 3$$

$$(5)\left(\frac{36}{5} - \frac{18}{5}y + 5y\right) = 3(5)$$

$$36 - 18y + 25y = 15$$

$$7y = 15 - 36$$

$$7y = -21$$

$$\frac{7y}{7} = \frac{-21}{7}$$

$$y = -3$$

Step 4. Substitute y = -3 into one of the original equations and solve for x.

$$5x + 6y = 12$$

$$5x + 6(-3) = 12$$

$$5x - 18 = 12$$

$$5x = 12 + 18$$

$$5x = 30$$

$$\frac{5x}{5} = \frac{30}{5}$$

$$x = 6$$

Step 5. Check the solution by substituting x = 6 and y = -3 into the other original equation.

$$3x + 5y = 3$$
$$3(6) + 5(-3) = 3$$
$$18 - 15 = 3$$
$$3 = 3$$

Thus, the solution checks.

Systems of two equations involving two unknowns can also be solved by comparison.

Step 1. Solve each equation for the same unknown in terms of the other unknown.

Step 2. Set the two expressions obtained equal to each other.

Step 3. Solve the resulting equation for the one remaining unknown.

Step 4. Find the value of the other unknown by substituting the value of the first unknown into one of the original equations.

Step 5. Check the solution by substituting the values of the two unknowns into the other original equation.

Example:

Solve the following system of equations by comparison.

$$5x + 6y = 12$$
$$3x + 5y = 3$$

Solution:

Step 1. Solve both equations for x.

$$5x + 6y = 12$$

$$5x = 12 - 6y$$

$$\frac{5x}{5} = \frac{12 - 6y}{5}$$

$$x = \frac{12 - 6y}{5}$$

$$3x + 5y = 3$$

$$3x = 3 - 5y$$

$$\frac{3x}{3} = \frac{3 - 5y}{3}$$

$$x = \frac{3 - 5y}{3}$$

Step 2. Set the two values for *x* equal to each other.

$$\frac{12 - 6y}{5} = \frac{3 - 5y}{3}$$

Step 3. Solve the resulting equation for *y*.

$$\frac{12 - 6y}{5} = \frac{3 - 5y}{3}$$

$$(3)(5)\ \frac{12 - 6y}{5} = \frac{3 - 5y}{3}\ (3)(5)$$

$$3(12 - 6y) = 5(3 - 5y)$$

$$36 - 18y = 15 - 25y$$

$$25y - 18y = 15 - 36$$

$$7y = -21$$

$$\frac{7y}{7} = \frac{-21}{7}$$

$$y = -3$$

Step 4. Substitute $y = -3$ into one of the original equations and solve for *x*.

$$5x + 6y = 12$$

$$5x + 6(-3) = 12$$

$$5x - 18 = 12$$

$$5x = 12 + 18$$

$$5x = 30$$

$$\frac{5x}{5} = \frac{30}{5}$$

$$x = 6$$

Step 5. Check the solution by substituting $x = 6$ and $y = -3$ into the other original equation.

$$3x + 5y = 3$$

$$3(6) + 5(-3) = 3$$

$$18 - 15 = 3$$

$$3 = 3$$

Thus, the solution checks.

Quite often, when more than one unknown exists in a problem, the end result of the equations expressing the problem is a set of simultaneous equations showing the relationship of one of the unknowns to the other unknowns.

Example:

Solve the following simultaneous equations by substitution.

$$3x + 4y = 6 \qquad\qquad 5x + 3y = -1$$

Solution:

Solve for x:

$$3x = 6 - 4y$$

$$x = 2 - \frac{4y}{3}$$

Substitute the value for x into the other equation:

$$5\left(2 - \frac{4y}{3}\right) + 3y = -1$$

$$10 - \frac{20y}{3} + 3y = -1$$

$$10 - \frac{20y}{3} + \frac{9y}{3} = -1$$

$$10 - \frac{11y}{3} = -1$$

$$-\frac{11y}{3} = -11$$

$$y = 3$$

Substitute $y = 3$ into the first equation:

$$3x + 4(3) = 6$$

$$3x = -6$$

$$x = -2$$

Check the solution by substituting $x = -2$ and $y = 3$ into the original equations.

$3x + 4y = 6$	$5x + 3y = -1$
$3(-2) + 4(3) = 6$	$5(-2) + 3(3) = -1$
$-6 + 12 = 6$	$-10 + 9 = -1$
$6 = 6$	$-1 = -1$

Thus, the solution checks.

Summary

The important information in this chapter is summarized below.

Simultaneous Equations Summary

There are three methods used when solving simultaneous equations:

- Addition or subtraction
- Substitution
- Comparison

WORD PROBLEMS

This chapter covers ways of setting up word problems and solving for the unknowns.

EO 1.5 Given a word problem, write equations and SOLVE for the unknown.

Basic Approach to Solving Algebraic Word Problems

Algebra is used to solve problems in science, industry, business, and the home. Algebraic equations can be used to describe laws of motion, pressures of gases, electric circuits, and nuclear facility operations. They can be applied to problems about the ages of people, the cost of articles, football scores, and other everyday matters. The basic approach to solving problems in these apparently dissimilar fields is the same. First, condense the available information into algebraic equations, and, second, solve the equations. Of these two basic steps, the first is frequently the most difficult to master because there are no clearly defined rules such as those that exist for solving equations.

Algebraic word problems should not be read with the objective of immediately determining the answer because only in the simpler problems is this possible. Word problems should be initially read to identify what answer is asked for and to determine which quantity or quantities, if known, will give this answer. All of these quantities are called the unknowns in the problem. Recognizing all of the unknowns and writing algebraic expressions to describe them is often the most difficult part of solving word problems. Quite often, it is possible to identify and express the unknowns in several different ways and still solve the problem. Just as often, it is possible to identify and express the unknowns in several ways that appear different but are actually the same relationship.

In writing algebraic expressions for the various quantities given in word problems, it is helpful to look for certain words that indicate mathematical operations. The words "sum" and "total" signify addition; the word "difference" signifies subtraction; the words "product," "times," and "multiples of" signify multiplication; the words "quotient," "divided by," "per," and "ratio" signify division; and the words "same as" and "equal to" signify equality. When quantities are connected by these words and others like them, these quantities can be written as algebraic expressions.

Sometimes you may want to write equations initially using words. For example, Bob is 30 years older than Joe. Express Bob’s age in terms of Joe’s.

Bob’s age = Joe’s age plus 30 years

If we let Bob's age be represented by the symbol B and Joe's age by the symbol J, this becomes

$$B = J + 30 \text{ years}$$

Examples:

Equations:

1. The total electrical output of one nuclear facility is 200 megawatts more than that of another nuclear facility.

   Let L be the output of the larger facility and S the capacity of the smaller facility. The statement above written in equation form becomes $L = 200MW + S$.

2. The flow in one branch of a piping system is one-third that in the other branch.

   If B is the flow in the branch with more flow, and b is the flow in the smaller branch, this statement becomes the equation $b = \frac{1}{3}\text{B}$.

3. A man is three times as old as his son was four years ago.

   Let M = man's age and S = son's age. Then $M = 3\,(S\text{-}4)$.

4. A car travels in one hour 40 miles less than twice as far as it travels in the next hour.

   Let x_1 be the distance it travels the first hour and x_2 the distance it travels the second then, $x_1 = (2)\,(x_2)\,\text{-}40$.

Steps for Solving Algebraic Word Problems

Algebraic word problems can involve any number of unknowns, and they can require any number of equations to solve. However, regardless of the number of unknowns or equations involved, the basic approach to solving these problems is the same. First, condense the available information into algebraic equations, and, second, solve the equations. The most straightforward type of algebraic word problems are those that require only one equation to solve. These problems are solved using five basic steps.

Step 1. Let some letter, such as x, represent one of the unknowns.

Step 2. Express the other unknowns in terms of x using the information given in the problem.

Step 3. Write an equation that says in symbols exactly what the problem says in words.

Step 4. Solve the equation.

Step 5. Check the answer to see that it satisfies the conditions stated in the problem.

Example 1:

What are the capacities of two water storage tanks in a nuclear facility if one holds 9 gallons less than three times the other, and their total capacity is 63 gallons?

Solution:

Step 1. Let x = Capacity of the Smaller Tank

Step 2. Then, $3x - 9$ = Capacity of the Larger Tank

Step 3. Total Capacity = Capacity of the Smaller Tank + Capacity of the Larger Tank

$$63 = x + (3x - 9)$$

Step 4. Solving for x:

$$x + (3x - 9) = 63$$

$$4x - 9 = 63$$

$$4x = 63 + 9$$

$$4x = 72$$

$$x = 18$$

Solving for the other unknown:

$$3x - 9 = 3(18) - 9$$

$$3x - 9 = 54 - 9$$

$$3x - 9 = 45$$

Answer: Capacity of the Smaller Tank = 18 gallons
Capacity of the Larger Tank = 45 gallons

Step 5. The larger tank holds 9 gallons less than three times the smaller tank.

$$3(18) - 9 = 54 - 9 = 45$$

The total capacity of the two tanks is 63 gallons.

$$18 + 45 = 63$$

Thus, the answers check.

Example 2:

A utility has three nuclear facilities that supply a total of 600 megawatts (Mw) of electricity to a particular area. The largest facility has a total electrical output three times that of the smallest facility. The third facility has an output that is 50 Mw more than half that of the largest facility. What is the electrical output of each of the three facilities?

Solution:

Step 1. Let x = Electrical Output of the Smallest Facility.

Step 2. Then,

$3x$ = Electrical Output of the Largest Facility,

and,

$\frac{3x}{2} + 50$ = Electrical Output of the Third Facility.

Step 3. Total Electrical Output = Sum of the Electrical Outputs of the Three Facilities.

$$600 = x + 3x + \frac{3x}{2} + 50$$

Step 4. Solving for x:

$$x + 3x + \frac{3x}{2} + 50 = 600$$

$$\frac{2x}{2} + \frac{6x}{2} + \frac{3x}{2} = 600 - 50$$

$$\frac{11x}{2} = 550$$

$$11x = 1100$$

$$x = 100$$

Solving for the other unknowns:

$$3x = 3(100)$$

$$3x = 300$$

$$\frac{1}{2}(3x) + 50 = \frac{1}{2}(300) + 50$$

$$\frac{1}{2}(3x) + 50 = 150 + 50$$

$$\frac{1}{2}(3x) + 50 = 200$$

Answers: Electrical Output of the Smallest Facility = 100 Mw
Electrical Output of the Largest Facility = 300 Mw
Electrical Output of the Third Facility = 200 Mw

Step 5. The largest facility has a total electrical output three times that of the smallest facility.

$$3(100) = 300$$

The other facility has an output which is 50 Mw more than half that of the largest facility.

$$\frac{1}{2}(300) + 50 = 150 + 50 = 200$$

The total output of the three facilities is 600 Mw.

$$100 + 200 + 300 = 600$$

Thus, the answers check.

Example 3:

The winning team in a football game scored 7 points less than twice the score of the losing team. If the total score of both teams was 35 points, what was the final score?

Solution:

Step 1. Let x = Winning Team's Score

Step 2. Then, $\frac{1}{2}(x + 7)$ = Losing Team's Score

Step 3. Total Score = Winning Team's Score + Losing Team's Score

$$35 = x + \frac{1}{2}(x + 7)$$

Step 4. Solving for x:

$$x + \frac{1}{2}(x + 7) = 35$$

$$2x + x + 7 = 70$$

$$3x = 70 - 7$$

$$3x = 63$$

$$x = 21 \text{ points}$$

Solving for the other unknowns:

$$\frac{1}{2}(x + 7) = \frac{1}{2}(21 + 7)$$

$$\frac{1}{2}(x + 7) = \frac{1}{2}(28)$$

$$\frac{1}{2}(x + 7) = 14 \text{ points}$$

Answers: Winning Team's Score = 21 points
Losing Team's Score = 14 points

Step 5. The winning team's score is 7 points less than twice the score of the losing team.

$$2(14) - 7 = 28 - 7 = 21 \text{ points}$$

The total score of both teams is 35 points.

$$21 + 14 = 35 \text{ points}$$

Thus, the answers check.

Example 4:

A man is 21 years older than his son. Five years ago he was four times as old as his son. How old is each now?

Solution:

Step 1. Let x = Son's Age Now

Step 2. Then,

$x + 21$ = Father's Age Now
$x - 5$ = Son's Age Five Years Ago
$(x + 21) - 5$ = Father's Age Five Years Ago

Step 3. Five years ago the father was four times as old as his son.

$$(x + 21) - 5 = 4(x - 5)$$

Step 4.

$$\begin{aligned} (x + 21) - 5 &= 4(x - 5) \\ x + 16 &= 4x - 20 \\ x - 4x &= -20 - 16 \\ -3x &= -36 \\ x &= 12 \text{ years} \end{aligned}$$

Solving for the other unknowns:

$$\begin{aligned} x + 21 &= 12 + 21 \\ x + 21 &= 33 \text{ years} \end{aligned}$$

Answers: Son's Age Now = 12 years
Father's Age Now = 33 years

Step 5. The man is 21 years older than his son.

$$12 + 21 = 33 \text{ years}$$

Five years ago he was four times as old as his son.

$$33 - 5 = 28 = 4(12 - 5) = 4 \text{ x } 7$$

Thus, the answers check.

Word Problems Involving Money

The five basic steps for solving algebraic word problems can be used for solving word problems involving money. Writing algebraic expressions for these problems depends on the general relationship between the total value and the unit value of money. The total value of a collection of money or a collection of items with a certain monetary value equals the sum of the numbers of items each multiplied by their unit values. Thus, the total value of five pennies, three nickels, four dimes, and two quarters is found by solving the following equation:

$$x = 5(\$0.01) + 3(\$0.05) + 4(\$.10) + 2(\$0.25)$$

$$x = \$0.05 + \$0.15 + \$0.40 + \$0.50$$

$$x = \$1.10$$

The total value of 25 tickets worth \$1.50 each and 30 tickets worth \$0.75 each is 25(\$1.50) + 30(\$0.75) which equals \$37.50 + \$22.50 or \$60.00. Algebraic word problems involving money are solved using this general relationship following the same five basic steps for solving any algebraic word problems.

Example 1:

> The promoter of a track meet engages a 6,000 seat armory. He wants to gross \$15,000. The price of children's tickets is to be one-half the price of adults' tickets. If one-third of the crowd is children, what should be the price of tickets, assuming capacity attendance?

Solution:

Step 1. Let x = Price of an Adult Ticket (in dollars)

Step 2. Then,

$\frac{x}{2}$ = Price of a Child's Ticket (in dollars)

$\frac{1}{3}(6{,}000) = 2{,}000$ = Number of Children's Tickets

$6{,}000 - 2{,}000 = 4{,}000$ = Number of Adults' Tickets

Step 3. Gross Income = (Number of Children's Tickets times their Unit Price) + (Number of Adults' Tickets times their Unit Price)

$$\$15{,}000 = 2{,}000\left(\frac{x}{2}\right) + 4{,}000(x)$$

Step 4. Solving for x:

$$15{,}000 = 2{,}000\left(\frac{x}{2}\right) + 4{,}000(x)$$

$$15{,}000 = 1{,}000x + 4{,}000x$$

$$15{,}000 = 5{,}000x$$

$$x = \$3.00$$

solving for the other unknown:

$$\frac{x}{2} = \text{Price of a Child's Ticket (in dollars)}$$

$$\frac{x}{2} = \frac{\$3.00}{2}$$

$$\frac{x}{2} = \$1.50$$

Answers: Price of Adults' Tickets = \$3.00
Price of Children's Tickets = \$1.50

Step 5. The price of children's tickets is one-half the price of adults' tickets.

$$\frac{1}{2}(\$3.00) = \$1.50$$

The gross is \$15,000.

4,000(\$3.00) + 2,000(\$1.50) = \$12,000 + \$3,000 = \$15,000

Thus, the answers check.

Example 2:

A collection of coins consists of nickels, dimes, and quarters. The number of quarters is twice the number of nickels, and the number of dimes is five more than the number of nickels. If the total amount of money is \$5.05, how many of each type of coin are in the collection?

Solution:

Step 1. Let x = Number of Nickels

Step 2. Then,

$2x$ = Number of Quarters
$x + 5$ = Number of Dimes

Step 3. Total Value = (Number of Nickels)(Value of a Nickel) + (Number of Dimes)(Value of a Dime) + (Number of Quarters)(Value of a Quarter)

$\$5.05 = (x)(\$0.05) + (x + 5)(\$0.10) + (2x)(\$0.25)$

Step 4. Solving for x:

$\$5.05 = (x)(\$0.05) + (x + 5)(\$0.10) + (2x)(\$0.25)$

$\$5.05 = \$0.05x + \$0.10x + \$0.50 + \$0.50x$

$\$5.05 = \$0.65x + \$0.50$

$\$0.65x = \$5.05 - \$0.50$

$\$0.65x = \$4.55$

$$x = \frac{\$4.55}{\$0.65}$$

$$x = 7$$

Solving for the other unknowns:

$$2x = 2(7)$$

$$2x = 14$$

$$x + 5 = 7 + 5$$

$$x + 5 = 12$$

Answers: Number of Nickels = 7

Number of Dimes = 12

Number of Quarters = 14

Step 5. The number of quarters is twice the number of nickels.

$$2(7) = 14$$

The number of dimes is five more than the number of nickels.

$$7 + 5 = 12$$

The total value is \$5.05.

7(\$0.05) + 12(\$0.10) + 14(\$0.25) =

\$0.35 + \$1.20 + \$3.50 = \$5.05

Thus, the answers check.

Problems Involving Motion

Many algebraic word problems involve fundamental physical relationships. Among the most common are problems involving motion. For example, the definition of speed is distance traveled divided by the time it takes. $V_{\text{ave}} = \dfrac{\text{distance}}{\text{time}} = \dfrac{d}{t}$ or multiplying both sides by t, $d = V_{\text{ave}} \times t$. For example, if a car travels at 50 miles per hour for 2 hours, the distance traveled equals (50 mi/hr)(2 hr) or 100 miles. This relationship applies for constant velocity motion only. In practice, it is applied more generally by using an average speed or average rate of travel for the time involved. The distance traveled is often represented by s; the average speed or average rate of travel, also called the average velocity, by v_{av}; and the time of travel by t.

$$s = v_{\text{av}}t \tag{2-13}$$

This same basic physical relationship can be written in two other forms, obtained by dividing both sides of the equation by v_{av} or by t.

$$t = \frac{s}{v_{av}} \tag{2-14}$$

$$v_{av} = \frac{s}{t} \tag{2-15}$$

Example 1:

How far can a car traveling at a rate of 52 miles per hour travel in 2½ hours?

Solution:

Using Equation 2-13:

$$s = v_{\text{av}}t$$

$$s = (52 \text{ miles/hour})(2\tfrac{1}{2} \text{ hours})$$

$$s = 130 \text{ miles}$$

Example 2:

How long does it take a plane traveling at 650 miles per hour to go 1430 miles?

Solution:

Using Equation 2-14:

$$t = \frac{s}{v_{av}}$$

$$t = \frac{1430 \text{ miles}}{650 \frac{\text{miles}}{\text{hour}}}$$

$$t = 2.2 \text{ hours}$$

Example 3:

What is the average speed of a train that completes a 450-mile trip in 5 hours?

Solution:

Using Equation 2-15:

$$v_{av} = \frac{s}{t}$$

$$v_{av} = \frac{450 \text{ miles}}{5 \text{ hours}}$$

$$v_{av} = 90 \text{ miles/hour}$$

Algebraic word problems involving motion are solved using the general relationship among distance, time, and average velocity following the same five basic steps for solving any algebraic word problem.

Example 1:

A plane flying at 525 miles per hour completes a trip in 2 hours less than another plane flying at 350 miles per hour. What is the distance traveled?

Solution:

Step 1. Let x = Distance Traveled (in miles)

Step 2. Then, using Equation 2-14,

$\frac{x}{525}$ = Time Taken by Faster Plane (in hours)

$\frac{x}{350}$ = Time Taken by Slower Plane (in hours)

Step 3. Time Taken by Faster Plane = Time Taken by Slower Plane - 2 hours

$$\frac{x}{525} \text{ hours} = \frac{x}{350} \text{ hours} - 2 \text{ hours}$$

$$\frac{x}{525} = \frac{x}{350} - \frac{700}{350}$$

$$\frac{x}{525} = \frac{x - 700}{350}$$

$$(350)(525)\left(\frac{x}{525}\right) = \left(\frac{x - 700}{350}\right)(350)(525)$$

$$350x = 525 \cdot (x - 700)$$

$$350x = 525x - 367{,}500$$

$$350x - 525x = -367{,}500$$

$$-175x = -367{,}5000$$

$$\frac{-175x}{-175} = \frac{-367{,}500}{-175}$$

$$x = 2100 \text{ miles}$$

Solving for the other unknowns:

$$\frac{x}{525} = \text{Time Taken by Faster Plane (in hours)}$$

$$\frac{x}{525} = \frac{2100}{525}$$

$$\frac{x}{525} = 4 \text{ hours}$$

$$\frac{x}{350} = \text{Time Taken by Slower Plane (in hours)}$$

$$\frac{x}{350} = \frac{2100}{350}$$

$$\frac{x}{350} = 6 \text{ hours}$$

Answers: Distance Traveled = 2100 miles
Time Taken by Faster Plane = 4 hours
Time Taken by Slower Plane = 6 hours

Step 5. The faster plane takes 2 hours less to complete the trip than the slower plane.

6 hours - 2 hours = 4 hours

Thus, the answer checks.

Example 2:

It takes a man 4 hours to reach a destination 1325 miles from his home. He drives to the airport at an average speed of 50 miles per hour, and the average speed of his plane trip is 500 miles per hour. How far does he travel by each mode of transportation?

Solution:

Step 1. Let x = Distance Traveled by Car (in miles)

Step 2. Then,

1325 - x = Distance Traveled by Plane (in miles)
and, using Equation 2-14,

$$\frac{x}{50} = \text{Time Traveled by Car (in hours)}$$

$$\frac{1325 - x}{500} = \text{Time Traveled by Plane (in hours)}$$

Step 3. Total Time = (Time Traveled by Car) + (Time Traveled by Plane)

$$4 \text{ hours} = \frac{x}{50} \text{ hours} + \frac{1325 - x}{500} \text{ hours}$$

Step 4. Solving for x:

$$4 = \frac{x}{50} + \frac{1325 - x}{500}$$

$$4 = \frac{10x + 1325 - x}{500}$$

$$(500)4 = \frac{9x + 1325}{500}(500)$$

$$2000 = 9x + 1325$$

$$2000 - 1325 = 9x$$

$$685 = 9x$$

$$\frac{9x}{9} = \frac{675}{9}$$

$$x = 75 \text{ miles}$$

Solving for the other unknowns:

$$\frac{x}{50} = \text{Time Traveled by Car (in hours)}$$

$$\frac{x}{50} = \frac{75}{50}$$

$$\frac{x}{50} = 1\frac{1}{2} \text{ hours}$$

$$\frac{1325 - x}{500} = \text{Time Traveled by Plane (in hours)}$$

$$\frac{1325 - x}{500} = \frac{1324 - 75}{500}$$

$$\frac{1325 - x}{500} = \frac{1250}{500} = 2\frac{1}{2} \text{ hours}$$

$1325 - x$ = Distance Traveled by Plane (in miles)

$1325 - x = 1325 - 75$

$1325 - x = 1250$ miles

Answers: Distance Traveled by Car = 75 miles
Distance Traveled by Plane = 1250 miles

Step 5. The total distance traveled is 1325 miles.

75 miles + 1250 miles = 1325 miles

The average speed by car is 50 miles per hour.

$$\frac{75 \text{ miles}}{1\frac{1}{2} \text{ hours}} = 50 \text{ miles per hour}$$

The average speed by plane is 500 miles per hour.

$$\frac{1250 \text{ miles}}{2\frac{1}{2} \text{ hours}} = 500 \text{ miles per hour}$$

The total time traveling is 4 hours.

1½ hours + 2½ hours = 4 hours

Thus, the answers check.

Solving Word Problems Involving Quadratic Equations

Many algebraic word problems involve quadratic equations. Any time the algebraic expressions describing the relationships in the problem involve a quantity multiplied by itself, a quadratic equation must be used to solve the problem. The steps for solving word problems involving quadratic equations are the same as for solving word problems involving linear equations.

Example:

> A radiation control point is set up near a solid waste disposal facility. The pad on which the facility is set up measures 20 feet by 30 feet. If the health physicist sets up a controlled walkway around the pad that reduces the area by 264 square feet, how wide is the walkway?

Solution:

Step 1. Let x = Width of the Walkway

Step 2. Then,
$30 - 2x$ = Length of Reduced Pad
$20 - 2x$ = Width of Reduced Pad

Step 3. Area of Reduced Pad = (Length of Reduced Pad)(Width of Reduced Pad)

$$600 - 264 = (30 - 2x)(20 - 2x)$$

$$336 = 600 - 100x + 4x^2$$

Step 4. Solve this quadratic equation.

$$4x^2 - 100x + 264 = 0$$

Using the Quadratic Formula, substitute the coefficients for *a*, *b*, and *c* and solve for *x*.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(4)(264)}}{2(4)}$$

$$x = \frac{100 \pm \sqrt{10{,}000 - 4{,}224}}{8}$$

$$x = \frac{100 \pm \sqrt{5{,}776}}{8}$$

$$x = \frac{100 \pm 76}{8}$$

$$x = \frac{100 + 76}{8}, \frac{100 - 76}{8}$$

$$x = \frac{176}{8}, \frac{24}{8}$$

$$x = 22,\ 3$$

The two roots are $x = 22$ feet and $x = 3$ feet. Since $x = 22$ feet is not physically meaningful, the answer is $x = 3$ feet.

Step 5. Check the answer.

The area of the reduced area pad is 264 square feet less than the area of the original pad.

$$600 - 264 = (20 - 2x)(30 - 2x)$$

$$336 = [20 - 2(3)][30 - 2(3)]$$

$$336 = (20 - 6)(30 - 6)$$

$$336 = (14)(24)$$

$$336 = 336$$

Thus, the answer checks.

Summary

The important information from this chapter is summarized below.

Algebraic Word Problems Summary

Algebraic word problems can easily be solved by following these five basic steps:

Step 1. Let some letter, such as *x*, represent one of the unknowns.

Step 2. Express the other unknowns in terms of *x* using the information given in the problem.

Step 3. Write an equation that represents in symbols exactly what the problem states in words.

Step 4. Solve the equation.

Step 5. Check the answer to see that it satisfies the conditions stated in the problem.

LOGARITHMS

This chapter covers changing the base of a logarithm and solving problems with logarithms.

EO 1.6 STATE the definition of a logarithm.

EO 1.7 CALCULATE the logarithm of a number.

Calculator Usage, Special Keys

This chapter will require the use of certain keys on a calculator to perform the necessary calculations. An understanding of the functions of each key will make logarithms (logs) an easy task.

Common Logarithm key

log

This key when pressed will compute the common log (base 10) of the number *x* in the display, where *x* is greater than zero.

Natural Logarithm key

ln$x$

This key when pressed will compute the natural logarithm (base *e*) of the number *x* in the display, where *x* is greater than zero.

INV

This key when pressed before the log and ln keys will compute the anti-log of the number *x* in the display. When used with the log key it will raise 10 to the displayed power ($10^{7.12}$) and when used with the ln key will raise (*e*) to the displayed power (e^{-381}).

Introduction

Logarithms are exponents, as will be explained in the following sections. Before the advent of calculators, logarithms had great use in multiplying and dividing numbers with many digits since adding exponents was less work than multiplying numbers. Now they are important in nuclear work because many laws governing physical behavior are in exponential form. Examples are radioactive decay, gamma absorption, and reactor power changes on a stable period.

Definition

Any number (X) can be expressed by any other number b (except zero) raised to a power x; that is, there is always a value of x such that $X = b^x$. For example, if $X = 8$ and $b = 2$, $x = 3$. For $X = 8$ and $b = 4$, $8 = 4^x$ is satisfied if $x = 3/2$.

$$4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = 8$$

or

$$4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = 2^3 = 8$$

In the equation $X = b^x$, the exponent x is the logarithm of X to the base b. Stated in equation form, $x = \log_b X$, which reads x is the logarithm to the base b of X. In general terms, the logarithm of a number to a base b is the power to which base b must be raised to yield the number. The rules for logs are a direct consequence of the rules for exponents, since that is what logs are. In multiplication, for example, consider the product of two numbers X and Y. Expressing each as b raised to a power and using the rules for exponents:

$$XY = (b^x)\,(b^y) = b^{x+y}$$

Now, equating the $\log_b$ of the first and last terms, $\log_b XY = \log_b b^{x+y}$.

Since the exponent of the base b ($x+y$) is the logarithm to the base b, $\text{Log}_b\, b^{x+y} = x+y$.

$$\log_b XY = x+y$$

Similarily, since $X = b^x$ and $Y = b^y$, $\log_b X = x$ and $\log_b Y = y$. Substituting these into the previous equation,

$$\log_b XY = \log_b X + \log_b Y$$

Before the advent of hand-held calculators it was common to use logs for multiplication (and division) of numbers having many significant figures. First, logs for the numbers to be multiplied were obtained from tables. Then, the numbers were added, and this sum (logarithm of the product) was used to locate in the tables the number which had this log. This is the product of the two numbers. A slide rule is designed to add logarithms as numbers are multiplied.

Logarithms can easily be computed with the calculator using the keys identified earlier.

Examples:

$\log_2 8 = 3$ since $8 = 2^3$

$\log_{10} 0.01 = -2$ since $0.01 = 10^{-2}$

$\log_5 5 = 1$ since $5 = 5^1$

$\log_b 1 = 0$ since $1 = b$

From the above illustration, it is evident that a logarithm is an exponent. 3^4 is called the exponential form of the number 81. In logarithmic form, 3^4 would be expressed as $\log_3 81 = 4$, or the logarithm of 81 to the base 3 is 4. Note the symbol for taking the logarithm of the number 81 to a particular base 3, is $\log_3 81$, where the base is indicated by a small number written to the right and slightly below the symbol log.

Log Rules

Since logs are exponents, the rules governing logs are very similar to the laws of exponents.

The most common log rules are the following:

1. $\log_b (ABC) = \log_b A + \log_b B + \log_b C$

2. $\log_b (A/B) = \log_b A - \log_b B$

3. $\log_b (A^n) = n\log_b A$

4. $\log_b b = 1$

5. $\log_b \sqrt[n]{A} = \log_b A^{1/n} = (1/n)\log_b A$

6. $\log_b 1 = 0$

7. $\log_b (1/A) = \log_b 1 - \log_b A = -\log_b A$

Example 1: $y = \frac{1}{2}\, gt^2$ where $g = 32$

Solution:

$y = 16\ t^2$

Find y for $t = 10$ using logs.

$\log_{10} y = \log_{10} (16\ t^2)$

$\log_{10} y = \log_{10} 16 + \log_{10} t^2$

$\log_{10} y = \log_{10} 16 + (2 \log_{10} t)$

$\log_{10} y = 1.204 + 2 \log_{10} 10$

$\log_{10} y = 1.204 + 2 \text{ x } 1$

$\log_{10} y = 3.204$

but this means $10^{3.204} = y$

$y = 1600$

Example 2: Calculate $\log_{10} 2 - \log_{10} 3$.

Solution:

Rule 2. $\log_{10} (A/B)$: $\log_{10} A - \log_{10} B$

$\log_{10} 2 - \log_{10} 3$

$= \log_{10} (2/3)$

$= \log_{10} (.667)$

$= -0.176$

Example 3: Calculate $3\log_{10} 2$.

Solution:

Rule 3. $\log_b (A^n) = n\log_b A$

$3\log_{10} 2$

$= \log_{10} (2^3)$

$= \log_{10} 8$

$= 0.903$

Example 4: Calculate $4\log_{10} 10$.

Solution:

Rule 4. $\log_b b = 1$

$4\log_{10} 10$

$= 4(1)$

$= 4$

Example 5: Calculate $(1/3)\log_{10} 2$.

Solution:

Rule 5. $\log_b \sqrt[n]{A} = \log_b A^{1/n} = (1/n)\log_b A$

$(1/3)\log_{10} 2$

$= \log_{10} \sqrt[3]{2}$

$= \log_{10} 1.259$

$= 0.1003$

Example 6: Calculate $\log_{10} 1$.

Solution:

Rule 6. $\log_b 1 = 0$

$\log_{10} 1 = 0$

Example 7: Calculate $-\log_{10} 2$.

Solution:

Rule 7: $\log_b (1/A) = -\log_b A$

$-\log_{10} 2$

$= \log_{10} (1/2)$

$= -\log_{10} 0.5$

$= -0.3010$

Common and Natural Logarithms

In scientific and engineering practice, the natural system of logarithms uses the number 2.718281828459042. Since this number is frequently encountered, the letter *e* is used. Many natural occurrences can be expressed by exponential equations with *e* as the base. For example, the decay of radioactive isotopes can be expressed as a natural logarithm equation. These logarithmic expressions are called natural logs because *e* is the basis for many laws of nature.

The expression *ln* is used to represent a logarithm when *e* is the base. Therefore, the exponential equation is written as

$e^x = N.$

and the logarithm expression is written as

$\log_e N = x$ or $\ln N = x$.

As with base 10 logs (common logs), natural logs can be determined easily with the aid of a calculator.

Base 10 logs are often referred to as common logs. Since base 10 is the most widely used number base, the "10" from the designation $\log_{10}$ is often dropped. Therefore, any time "log" is used without a base specified, one should assume that base 10 is being used.

Anti-Logarithms

An anti-logarithm is the opposite of a logarithm. Thus, finding the anti-logarithm of a number is the same as finding the value for which the given number is the logarithm. If $\log_{10} X = 2$, then 2.0 is the power (exponent) to which one must raise the base 10 to obtain X, that is, $X = 10^{2.0} = 100$. The determination of an anti-log is the reverse process of finding a logarithm.

Example:

Multiply 38.79 and 6896 using logarithms.

Log 38.79 = 1.58872 Log 6896 = 3.83860

Add the logarithms to get 5.42732

Find the anti-log.

Anti-log 5.42732 = 2.675×10^5 = 267,500

Thus, 38.79 x 6896 = 2.675×10^5 = 267,500

Natural and Common Log Operations

The utilization of the log/ln can be seen by trying to solve the following equation algebraically. This equation cannot be solved by algebraic methods. The mechanism for solving this equation is as follows:

Using Common Logs

$$2^X = 7$$
$$\log 2^X = \log 7$$
$$X \log 2 = \log 7$$
$$X = \frac{\log 7}{\log 2} = \frac{0.8451}{0.3010} = 2.808$$

Using Natural Logs

$$2^X = 7$$
$$\ln 2^X = \ln 7$$
$$X \ln 2 = \ln 7$$
$$X = \frac{\ln 7}{\ln 2} = \frac{1.946}{0.693} = 2.808$$

How would you calculate x in the following equation?

$$\log x = 5$$

The easy way to solve this equation is to take the anti-log. As division is the reverse of multiplication, so anti-log is the reverse of log. To take the anti-log $\log_{10} x = 5$:

$$\text{anti-log } (\log X) = \text{anti-log } 5$$

$$x = \text{anti-log } 5$$

$$x = 100{,}000$$

This is accomplished on a calculator by pressing the 5, INV, then the LOG key. This causes the inverse of the log process.

Summary

The important information in this chapter is summarized below.

Logarithms Summary

- A number L is said to be the logarithm of a positive real number N to the base b (where b is real, positive, and not equal to 1), if L is the exponent to which b must be raised to obtain N, or the function can be expressed as

  $$L = \mathrm{Log}_b\ N$$

  for which the inverse is

  $$N = b^L$$

  Simply stated, the logarithm is the inverse of the exponential function.

- Product = base$^{\text{exponent}}$
- $\mathrm{Log}_{\text{base}}$product = exponent
- $\mathrm{Log}_b\ (ABC) = \log_b A + \log_b B + \log_b C$
- $\mathrm{Log}_b\ (A/B) = \log_b A - \log_b B$
- $\mathrm{Log}_b\ (A^n) = n\log_b A$
- $\mathrm{Log}_b \sqrt[n]{A} = \log_b A^{1/n} = (1/n)\log_b A$
- $\mathrm{Log}_b\ 1 = 0$
- $\mathrm{Log}_b\ (1/A) = \log_b 1 - \log_b A = -\log_b A$
- Common logs are base 10
- Natural logs are base e
- Anti-log is the opposite of a log

GRAPHING

This chapter covers graphing functions and linear equations using various types of graphing systems.

EO 1.8 STATE the definition of the following terms:
a. Ordinate
b. Abscissa

EO 1.9 Given a table of data, PLOT the data points on a cartesian coordinate graph.

EO 1.10 Given a table of data, PLOT the data points on a logarithmic coordinate graph.

EO 1.11 Given a table of data, PLOT the data points on the appropriate graphing system to obtain the specified curve.

EO 1.12 Obtain data from a given graph.

EO 1.13 Given the data, SOLVE for the unknown using a nomograph.

In work with physical systems, the relationship of one physical quantity to another is often of interest. For example, the power level of a nuclear reactor can be measured at any given time. However, this power level changes with time and is often monitored. One method of relating one physical quantity to another is to tabulate measurements. Thus, the power level of a nuclear reactor at specific times can be recorded in a log book. Although this method does provide information on the relationship between power level and time, it is a difficult method to use effectively. In particular, trends or changes are hard to visualize. Graphs often overcome these disadvantages. For this reason, graphs are widely used.

A graph is a pictorial representation of the relationship between two or more physical quantities. Graphs are used frequently both to present fundamental data on the behavior of physical systems and to monitor the operation of such systems. The basic principle of any graph is that distances are used to represent the magnitudes of numbers. The number line is the simplest type of graph. All numbers are represented as distances along the line. Positive numbers are located to the right of zero, and negative numbers are located to the left of zero.

The coordinate system of a graph is the framework upon which the graph is drawn. A coordinate system consists of numbered scales that give the base and the direction for measuring points on the graph. Any point on a graph can be specified by giving its coordinates. Coordinates describe the location of the point with respect to the scales of the coordinate system. There are several different coordinate systems commonly encountered.

The Cartesian Coordinate System

The Cartesian Coordinate System, also known as the rectangular coordinate system, consists of two number scales, called the x-axis (at y = 0) and the y-axis (at x = 0), that are perpendicular to each other. Each scale is a number line drawn to intersect the other at zero. The zero point is called the origin. The divisions along the scales may be any size, but each division must be equal. Figure 1 shows a rectangular coordinate system. The axes divide the coordinate system into four regions called quadrants. Quadrant I is the region above the x-axis and to the right of the y-axis. Quadrant II is the region above the x-axis and to the left of the y-axis. Quadrant III is the region below the x-axis and to the left of the y-axis. Quadrant IV is the region below the x-axis and to the right of the y-axis.

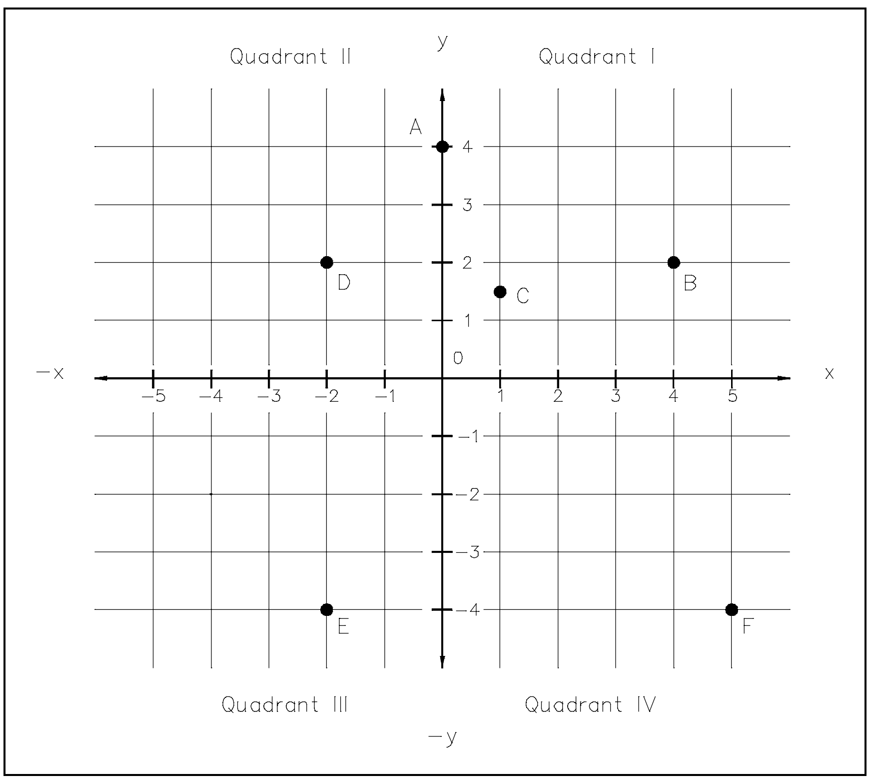


Figure 1 The Cartesian System

The use of a graph starts with the plotting of data points using the coordinate system. These data points are known as the **abscissa** and the **ordinate**. The abscissa, also known as the y-coordinate, is the distance along the y-axis. The ordinate, also known as the x-coordinate, is the distance along the x-axis. A point on a Cartesian coordinate graph is specified by giving its x-coordinate and its y-coordinate. Positive values of the x-coordinate are measured to the right, negative values to the left. Positive values of the y-coordinate are measured up, negative values down. For example, the x- and y-coordinates are both zero at the origin. The origin is denoted as (0,0), where the first zero refers to the value of the x-coordinate. Point A in Figure 1 is denoted as (0,4), since the value of the x-coordinate is zero, and the value of the y-coordinate is 4. In Quadrant I, every point has a positive x-coordinate and a positive y-coordinate. Point B in Figure 1 is located in Quadrant I and is denoted by (4,2). Fractional values of coordinates can also be shown. Point C in Figure 1 is denoted by (1,1.5). In Quadrant II, every point has a negative x-coordinate and a positive y-coordinate. Point D is denoted by (-2,2). In Quadrant III, every point has a negative x-coordinate and a negative y-coordinate. Point E is located in Quadrant III and is denoted by (-2,-4). In Quadrant IV, every point has a positive x-coordinate, but a negative y-coordinate. Point F is located in Quadrant IV and is denoted by (5,-4).

Cartesian Coordinate Graphs

The most common type of graph using the Cartesian Coordinate System is one in which all values of both the x-coordinate and the y-coordinate are positive. This corresponds to Quadrant I of a Cartesian coordinate graph. The relationship between two physical quantities is often shown on this type of rectangular plot. The x-axis and the y-axis must first be labeled to correspond to one of the physical quantities. The units of measurement along each axis must also be established. For example, to show the relationship between reactor power level and time, the x-axis can be used for time in minutes and the y-axis for the reactor power level as a percentage of full power level. Data points are plotted using the associated values of the two physical quantities.

Example: The temperature of water flowing in a high pressure line was measured at regular intervals. Plot the following recorded data on a Cartesian coordinate graph.

Time (min)	Temperature (°F)
0	400°
15	420°
30	440°
45	460°
60	480°
75	497°
90	497°
105	497°
120	497°

The first step is to label the x-axis and the y-axis. Let the x-axis be time in minutes and the y-axis be temperature in °F.

The next step is to establish the units of measurement along each axis. The x-axis must range from 0 to 120, the y-axis from 400 to 500.

The points are then plotted one by one. Figure 2 shows the resulting Cartesian coordinate graph.

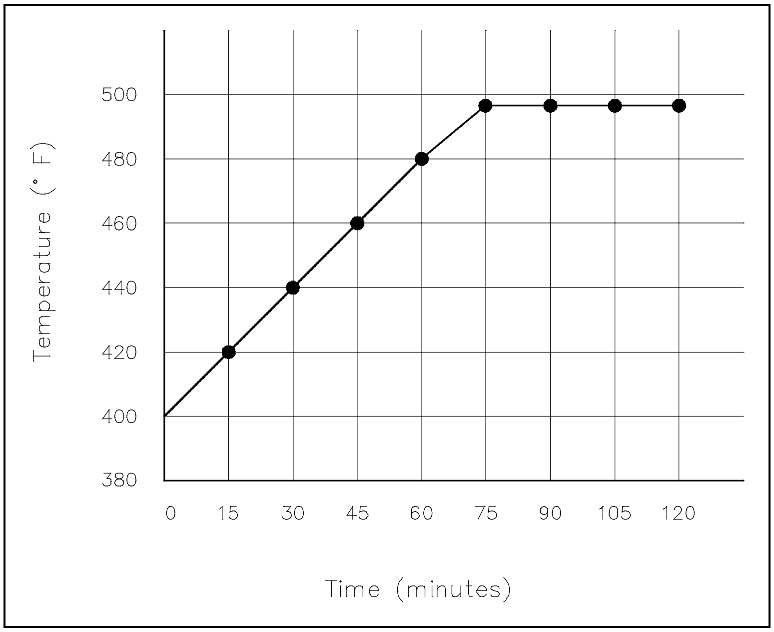


Figure 2 Cartesian Coordinate Graph of Temperature vs. Time

Example: The density of water was measured over a range of temperatures. Plot the following recorded data on a Cartesian coordinate graph.

Temperature (°C)	Density (g/ml)
40°	0.992
50°	0.988
60°	0.983
70°	0.978
80°	0.972
90°	0.965
100°	0.958

The first step is to label the x-axis and the y-axis. Let the x-axis be temperature in °C and the y-axis be density in g/ml.

The next step is to establish the units of measurement along each axis. The x-axis must range from approximately 40 to 100, the y-axis from 0.95 to 1.00.

The points are then plotted one by one. Figure 3 shows the resulting Cartesian coordinate graph.

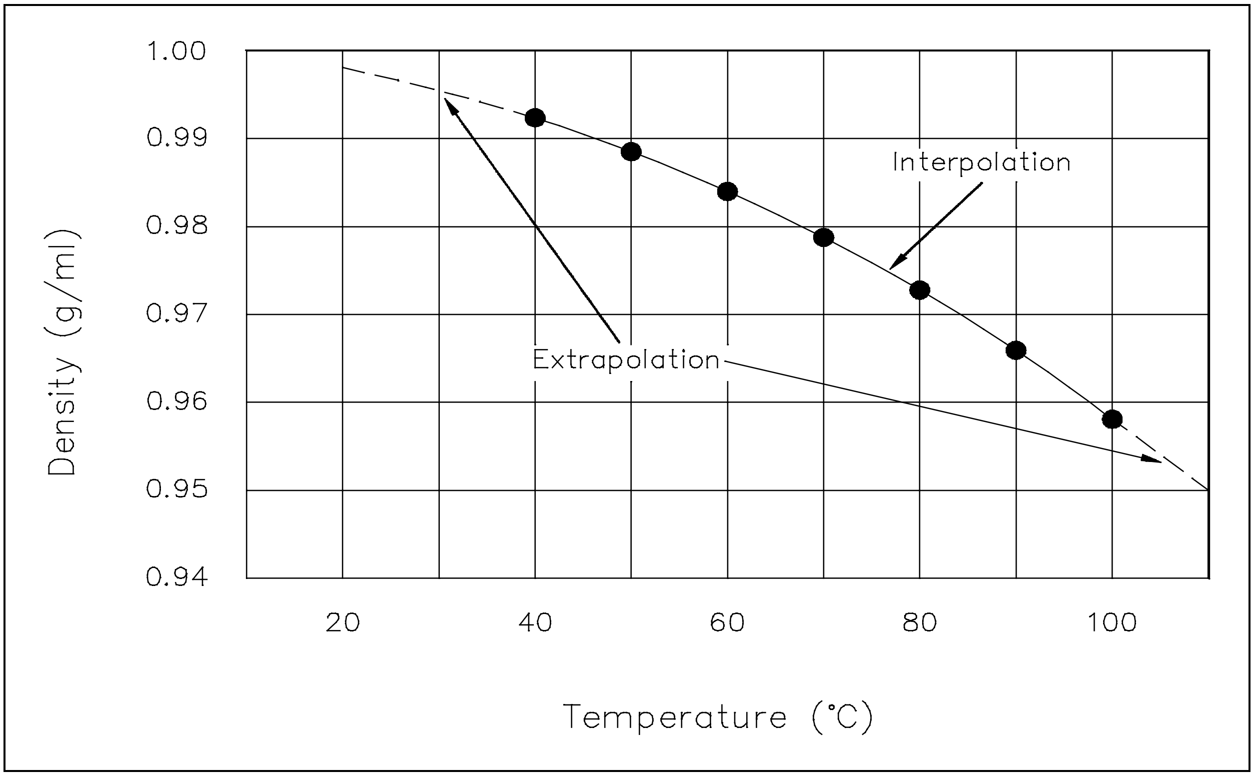


Figure 3 Cartesian Coordinate Graph of Density of Water vs. Temperature

Graphs are convenient because, at a single glance, the major features of the relationship between the two physical quantities plotted can be seen. In addition, if some previous knowledge of the physical system under consideration is available, the numerical value pairs of points can be connected by a straight line or a smooth curve. From these plots, the values at points not specifically measured or calculated can be obtained. In Figures 2 and 3, the data points have been connected by a straight line and a smooth curve, respectively. From these plots, the values at points not specifically plotted can be determined. For example, using Figure 3, the density of water at 65°C can be determined to be 0.98 g/ml. Because 65°C is within the scope of the available data, it is called an interpolated value. Also using Figure 3, the density of water at 101°C can be estimated to be 0.956 g/ml. Because 101°C is outside the scope of the available data, it is called an extrapolated value. Although the value of 0.956 g/ml appears reasonable, an important physical fact is absent and not predictable from the data given. Water boils at 100°C at atmospheric pressure. At temperatures above 100°C it is not a liquid, but a gas. Therefore, the value of 0.956 g/ml is of no significance except when the pressure is above atmospheric.

This illustrates the relative ease of interpolating and extrapolating using graphs. It also points out the precautions that must be taken, namely, interpolation and extrapolation should be done only if there is some prior knowledge of the system. This is particularly true for extrapolation where the available data is being extended into a region where unknown physical changes may take place.

Logarithmic Graphs

Frequently, the function to be plotted on a graph makes it convenient to use scales different from those used for the Cartesian coordinate graphs. Logarithmic graphs in which one or both of the scales are divided logarithmically are common. A semi-log plot is used when the function is an exponential, such as radioactive decay. A semi-log plot is obtained by using an ordinary linear scale for one axis and a logarithmic scale for the other axis. A log-log plot is used when the function is a power. A log-log plot is obtained by using logarithmic scales for both axes. Table 1 gives data on the amount of radioactive strontium 90 present as a function of time in years. Every twenty-five years one-half of the material decays. Figure 4 is a Cartesian coordinate graph of the data given in Table 1. It can be seen from Figure 4 that it is difficult to determine from this plot the amount of strontium 90 present after long periods of time such as 125 years, 150 years, or 175 years.

TABLE 1
Data on the Radioactive Decay of Strontium 90

Time (years)	Amount of Strontium 90 (grams)
0	100
25	50
50	25
75	12.5
100	6.25
125	3.125
150	1.5625
175	0.78125

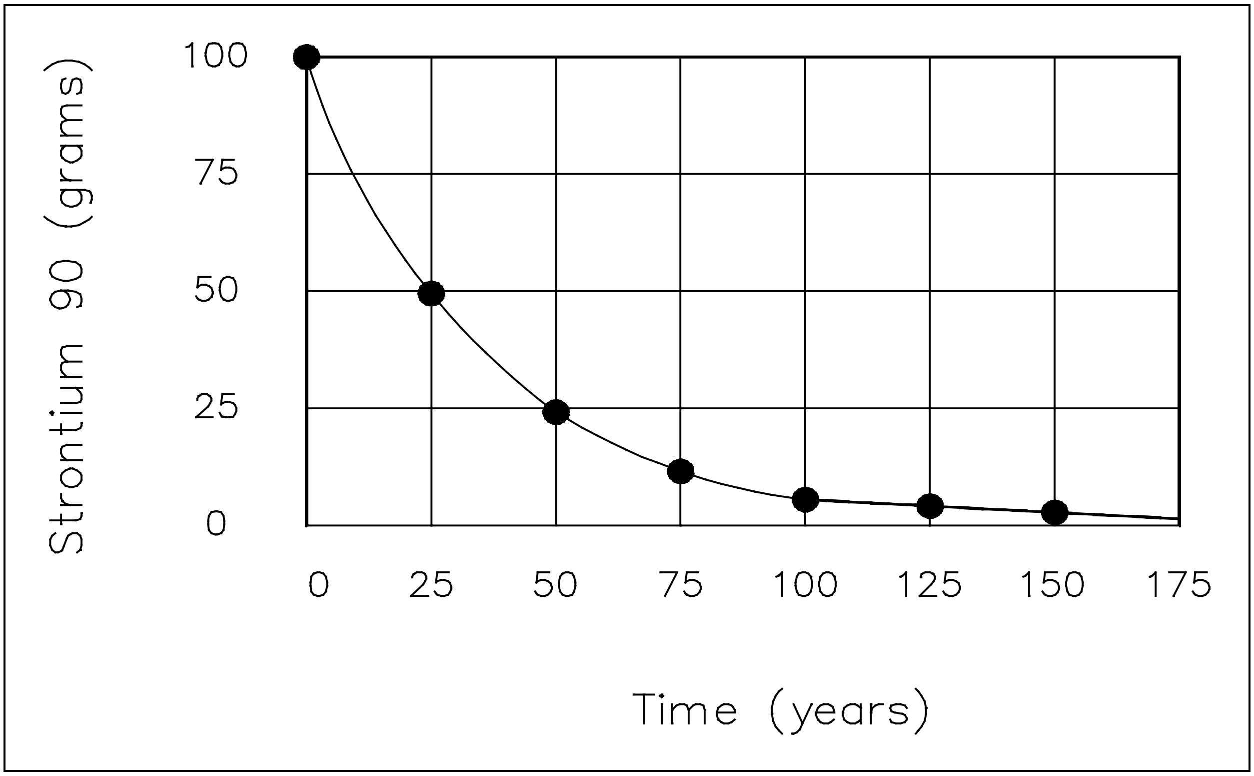


Figure 4 Cartesian Coordinate Plot of Radioactive Decay of Strontium 90

If the same data, the decay of strontium 90, is plotted on semi-log, the resulting plot (Figure 5) will be a straight line. This is because the decay of radioactive material is an exponential function. The resulting straight line of the semi-log plot allows a more accurate extrapolation or interpolation of the data than the curve obtained from the cartesian plot.

For graphs in which both of the quantities (x,y) vary as a power function, a log-log plot is convenient. A log-log plot is obtained by using logarithmic scales for both axes. Table 2 gives data on the frequency of electromagnetic radiation as a function of the wavelength of the radiation. Figure 6 is a log-log plot of the data given in Table 2.

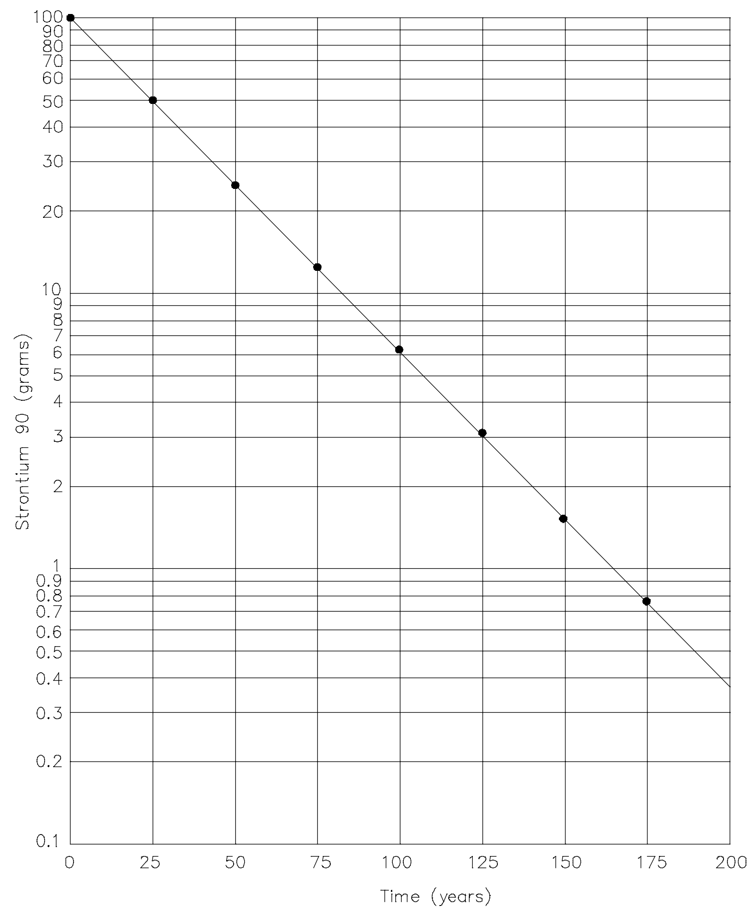


Figure 5 Semi-log Plot of Radioactive Decay of Strontium 90

TABLE 2
Data on Frequency vs. Wavelength of Electromagnetic Radiation

Wavelength (cm)	Frequency (s^{-1})
1.0×10^{-8}	3×10^{18}
0.5×10^{-7}	6×10^{17}
1.0×10^{-7}	3×10^{17}
0.5×10^{-6}	6×10^{16}
1.0×10^{-6}	3×10^{16}

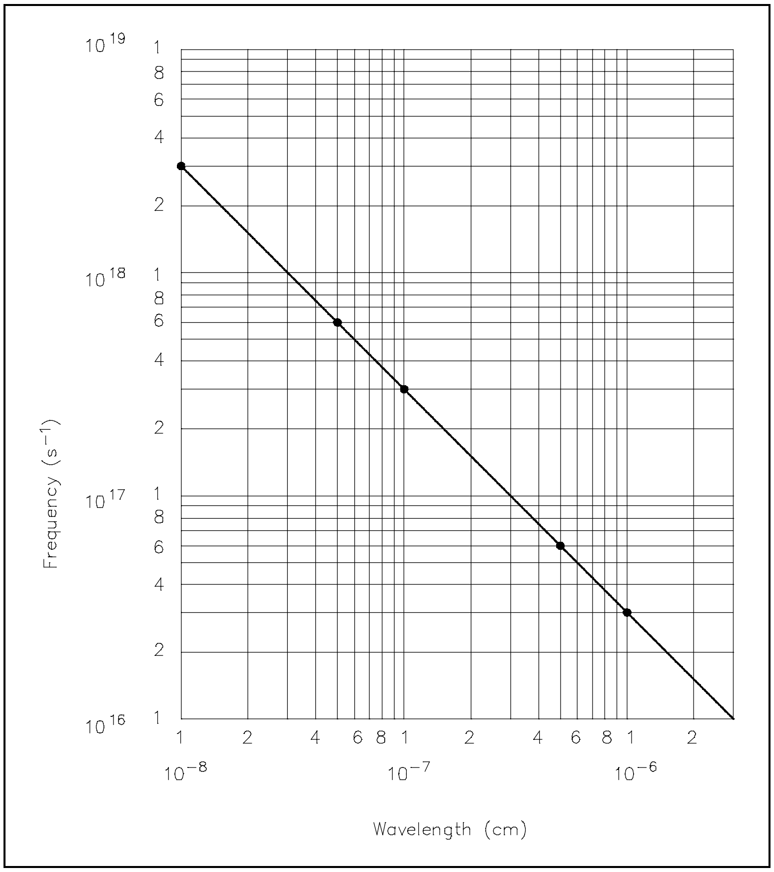


Figure 6 Log-Log Plot of Frequency vs. Wavelength of Electromagnetic Radiation

In summary, the type of coordinate system used to plot data, cartesian, semi-log, or log-log, should be based on the type of function to be graphed and the desired shape (curve or line) of the curve wanted.

- Cartesian system - Linear ($y = mx + b$) type functions when plotted will provide straight lines; exponential functions ($y = e^x$) will plot as curves.
- Semi-log system - Should not plot linear type functions on semi-log. Exponential functions, such as radioactive decay and reactor power equations when plotted will graph as straight lines.
- Log-log - Rarely used; used to plot power equations.

Graphing Equations

Algebraic equations involving two unknowns can readily be shown on a graph. Figure 7 shows a plot of the equation $x + y = 5$. The equation is solved for corresponding sets of values of x and y that satisfy the equation. Each of these points is plotted and the points connected. The graph of $x + y = 5$ is a straight line.

x	y
−3	8
−2	7
−1	6
0	5
1	4
2	3
3	2
4	1
5	0
6	−1
7	−2
8	−3

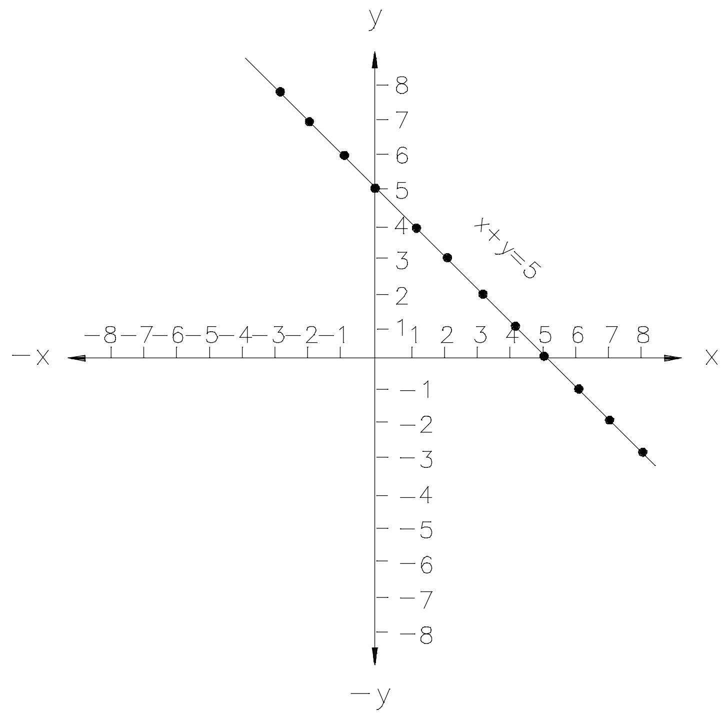


Figure 7 Plot of x + y = 5

The x-intercept of a line on a graph is defined as the value of the x-coordinate when the y-coordinate is zero. It is the value of x where the graph intercepts the x-axis. The y-intercept of a graph is defined as the value of the y-coordinate when the x-coordinate is zero. It is the value of y where the graph intercepts the y-axis. Thus, the x-intercept of the graph of $x + y = 5$ is +5. For a linear equation in the general form $ax + by = c$, the x-intercept and y-intercept can also be given in general form.

Any algebraic equation involving two unknowns of any function relating two physical quantities can be plotted on a Cartesian coordinate graph. Linear equations or linear functions plot as straight lines on Cartesian coordinate graphs. For example, $x + y = 5$ and $f(x) = 3x + 9$ plot as straight lines. Higher order equations or functions, such as quadratic equations or functions and exponential equations, can be plotted on Cartesian coordinate graphs. Figure 8 shows the shape of the graph of a typical quadratic equation or function. This shape is called a parabola. Figure 9 shows the shape of the graph of a typical exponential equation or function.

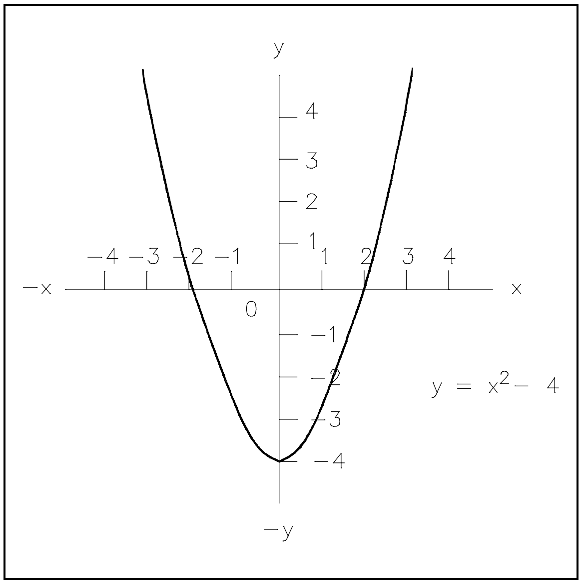


Figure 8 Cartesian Coordinate Graph of Quadratic Equation or Function

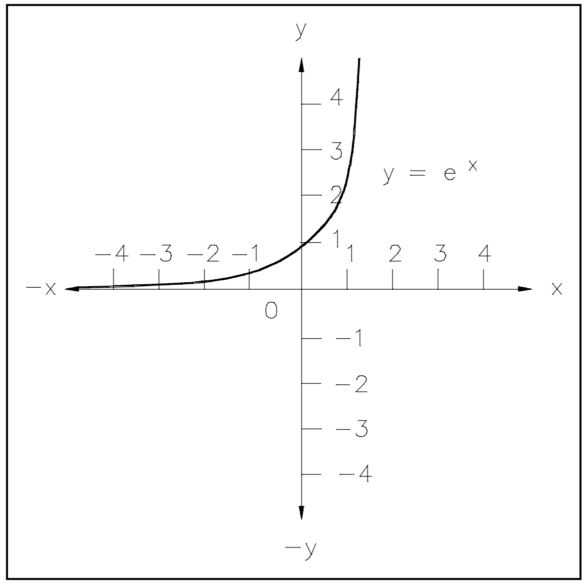


Figure 9 Cartesian Coordinate Graph of Exponential Equation or Function

Nomographs

A nomograph is a device used to relate the physical quantities in such a way that the value of an unknown quantity can be determined given the values of the other related quantities. Nomographs normally involve the relationship among three physical quantities. The scales are located in such a way that, when a straight line is drawn between the values of the known quantities on their respective scales, the line crosses the value of the unknown quantity on its scale. Figure 10 is a typical nomograph that relates the distance traveled, the average speed, and the time traveled. It should be noted that, as with any graphical representation, the values determined are only approximations.

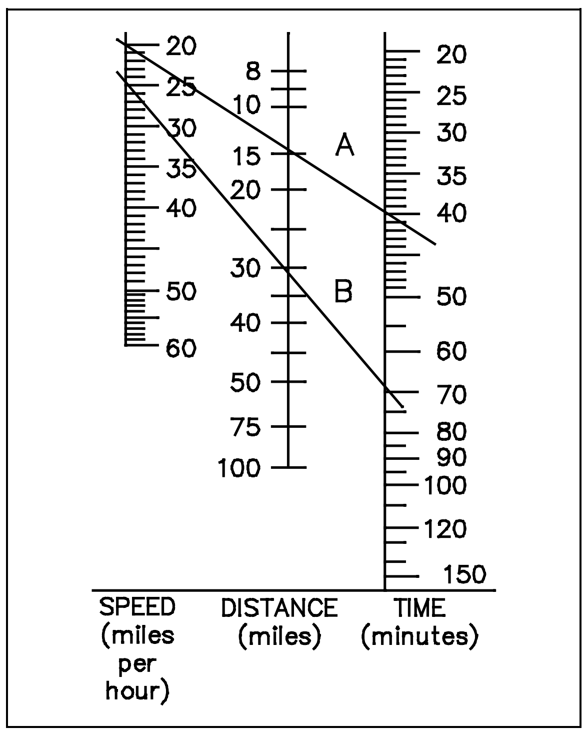


Figure 10 Typical Nomograph

Example:

Using Figure 10, find the distance traveled if the average speed is 20 mph and the time traveled is 40 minutes.

The line labeled A in Figure 10 connects 20 mph and 40 minutes. It passes through 14.5 miles.

Thus, the distance traveled is 14.5 miles.

Example:

Using Figure 10, find the time required to travel 31 miles at an average speed of 25 mph.

The line labeled B in Figure 10 connects 31 miles and 25 mph. It passes through 70 minutes.

Thus, the time required is 70 minutes.

Summary

The important information in this chapter is summarized below.

Graphing Summary

- Ordinate - x-coordinate
- Abscissa - y-coordinate
- Cartesian Coordinate System
    - Rectangular Coordinate System
    - Divided into four quadrants by x- and y-axis
- Logarithmic Coordinate System
    - One or both of the scales are divided logarithmically
    - Semi-log graphs contain linear x-axis and logarithmic y-axis
    - Log-log graphs contain logarithmic x- and y-axis
- Linear functions are usually plotted on Cartesian coordinate graph.
- Exponential functions ($y = e^x$) are usually plotted on semi-log graphs to provide a straight line instead of the resulting curve placed on a Cartesian coordinate graph.
- Power functions ($Y = ax^2$, $y = ax^3$, etc.) are usually plotted on log-log graphs.

SLOPES

This chapter covers determining and calculating the slope of a line.

EO 1.14 STATE the definition of the following terms:
a. Slope
b. Intercept

EO 1.15 Given the equation, CALCULATE the slope of a line.

EO 1.16 Given the graph, DETERMINE the slope of a line.

Many physical relationships in science and engineering may be expressed by plotting a straight line. The slope(m), or steepness, of a straight line tells us the amount one parameter changes for a certain amount of change in another parameter.

Slope

For a straight line, slope is equal to rise over run, or

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Consider the curve shown in Figure 11. Points $P1$ and $P2$ are any two different points on the line, and a right triangle is drawn whose legs are parallel to the coordinate axes. The length of the leg parallel to the x-axis is the difference between the x-coordinates of the two points and is called "Δx," read "delta x," or "the change in x." The leg parallel to the y-axis has length Δy, which is the difference between the y-coordinates. For example, consider the line containing points (1,3) and (3,7) in the second part of the figure. The difference between the x-coordinates is Δx = 3-1 = 2. The difference between the y-coordinates is Δy = 7-3 = 4. The ratio of the differences, $\Delta y/\Delta x$, is the slope, which in the preceding example is 4/2 or 2. It is important to notice that if other points had been chosen on the same line, the ratio $\Delta y/\Delta x$ would be the same, since the triangles are clearly similar. If the points (2,5) and (4,9) had been chosen, then $\Delta y/\Delta x$ = (9-5)/(4-2) = 2, which is the same number as before. Therefore, the ratio $\Delta y/\Delta x$ depends on the inclination of the line, m = rise [vertical (y-axis) change] ÷ run [horizontal (x-axis) change].

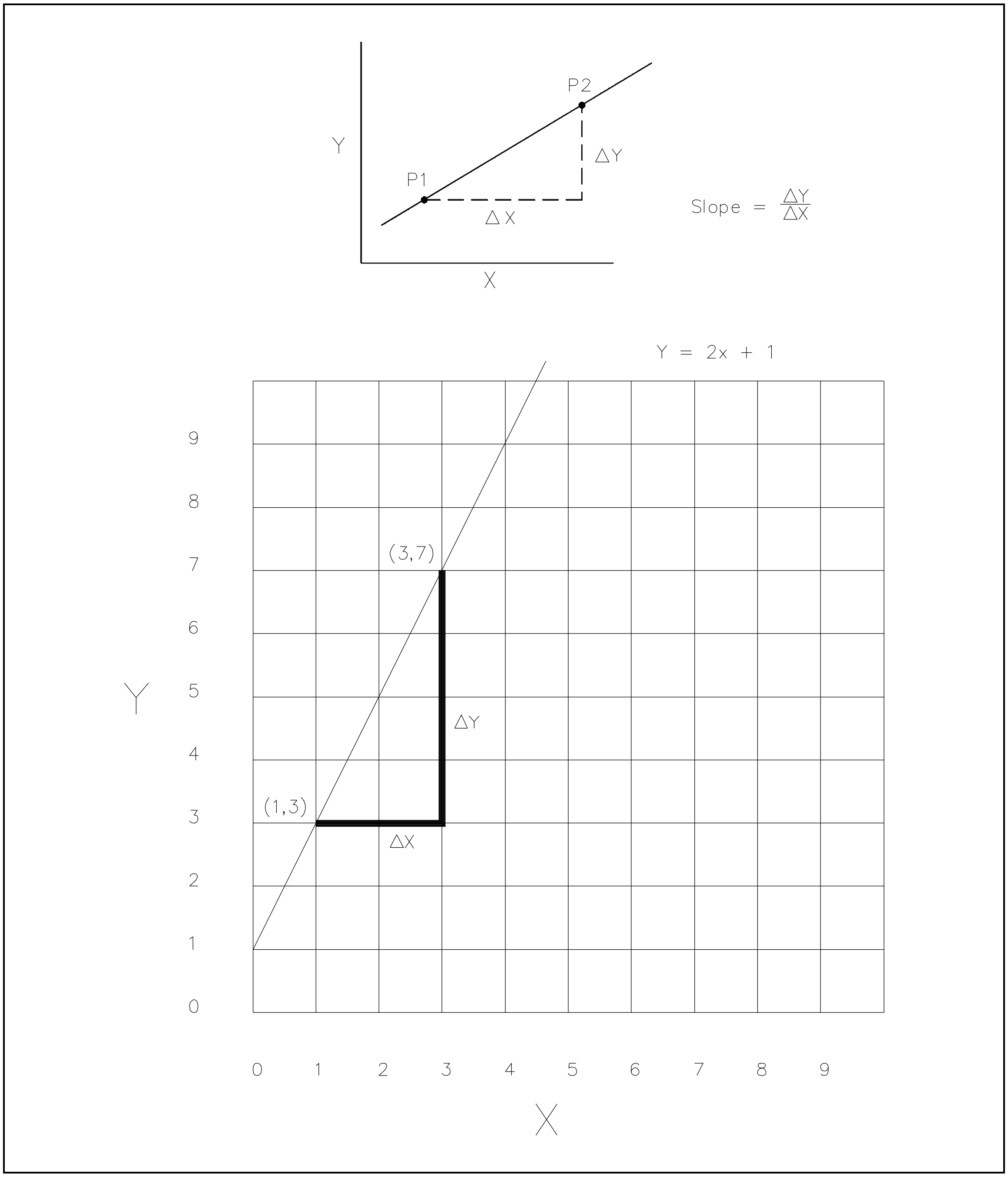


Figure 11 Slope

Since slope m is a measure of the steepness of a line, a slope has the following characteristics:

1. A horizontal line has zero slope.

2. A line that rises to the right has positive slope.

3. A line rising to the left has negative slope.

4. A vertical line has undefined slope because the calculation of the slope would involve division by zero. ($\Delta y/\Delta x$ approaches infinity as the slope approaches vertical.)

Example: What is the slope of the line passing through the points (20, 85) and (30, 125)?

Solution: $m = \dfrac{125 - 85}{30 - 20} = \dfrac{40}{10} = 4$

Given the coordinates of the y-intercept where the line crosses the y-axis [written (0, y)] and the equation of the line, determine the slope of the line.

The standard linear equation form is $y = mx + b$. If an equation is given in this standard form, m is the slope and b is the y coordinate for the y-intercept.

Example: Determine the slope of the line whose equation is $y = 2x + 3$ and whose y-intercept is (0,3).

Solution: $y = mx + b$

$y = 2x + 3$

$m = 2$

Example: Determine the slope of the line whose equation is $2x + 3y = 6$ and whose y-intercept is (0,2).

Solution: $y = mx + b$

$2x + 3y = 6$ Write in standard form.

$3y = 6 - 2x$

$3y = -2x + 6$

$$y = \frac{-2x + 6}{3}$$

$y = -2/3x + 2$

$m = -2/3$

Example:

Plot the graph of the following linear function. Determine the x-intercept, the y-intercept, and the slope.

$7x + 3y = 21$

Solution: $y = mx + b$

$y = (-7/3)x + 7$

x-intercept = 3
y-intercept = 7

Slope = -2.333

Summary

The important information in this chapter is summarized below.

Slopes Summary

For a straight line, slope is equal to rise over run, or

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

Since slope *m* is a measure of the steepness of a line, a slope has the following characteristics:

1. A horizontal line has zero slope.
2. A line that rises to the right of vertical has positive slope.
3. A line rising to the left of vertical has negative slope.
4. A vertical line has undefined slope because the calculation of the slope would involve division by zero ($\Delta y/\Delta x$ approaches infinity as the slope approaches vertical).

INTERPOLATION AND EXTRAPOLATION

This chapter covers the use of interpolation and extrapolation to solve for unknowns on various types of graphs.

EO 1.17 Given a graph, SOLVE for the unknown using extrapolation.

EO 1.18 Given a graph, SOLVE for the unknown using interpolation.

Definitions

Interpolation Interpolation is the process of obtaining a value from a graph or table that is located between major points given, or between data points plotted. A ratio process is usually used to obtain the value.

Extrapolation Extrapolation is the process of obtaining a value from a chart or graph that extends beyond the given data. The "trend" of the data is extended past the last point given and an estimate made of the value.

Interpolation and Extrapolation

Developing a curve from a set of data provides the student with the opportunity to interpolate between given data points. Using the curve in the following example, the value of the dependent variable at 4.5 can be estimated by interpolating on the curve between the two data points given, resulting in the value of 32. Note that the interpolation is the process of obtaining a value on the plotted graph that lies between two given data points. Extrapolation is the process in which information is gained from plotted data by extending the data curve beyond the points of given data (using the basic shape of the curve as a guide), and then estimating the value of a given point by using the extended (extrapolated) curve as the source. The above principles are illustrated in the example that follows.

Example:

Given equation $y = x^2 + 2x + 3$:

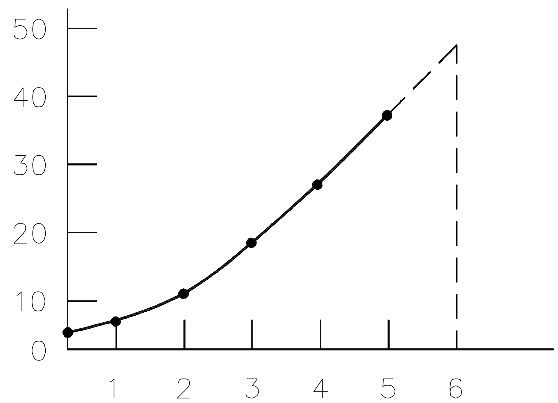
Plot the curve for x from 0 to 5.

Extrapolate the curve and give the value of y at $x = 6$.

Put 6 into the equation evaluating y, then compare the values.

Interpolate the curve at $x = 4.5$.

Put 4.5 into the equation evaluating y, then compare the values.



Extrapolating $x = 6$ gives a value of $y = 48$.

Using the equation, the actual value of y is 51.

Interpolating $x = 4.5$ gives a value of $y = 32$.

Using the equation, the actual value of y is 32.25.

Summary

The important information in this chapter is summarized below.

Interpolation and Extrapolation Summary

Interpolation	Interpolation is the process of obtaining a value from a graph or table that is located between major points given, or between data points plotted. A ratio process is usually used to obtain the value.
Extrapolation	Extrapolation is the process of obtaining a value from a chart or graph that extends beyond the given data. The "trend" of the data is extended past the last point given and an estimate made of the value.