MATHEMATICS
Module 5
Higher Concepts of Mathematics

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TERMINAL OBJECTIVE

1.0 **SOLVE** problems involving probability and simple statistics.

ENABLING OBJECTIVES

1.1 **STATE** the definition of the following statistical terms:
   a. Mean
   b. Variance
   c. Mean variance

1.2 **CALCULATE** the mathematical mean of a given set of data.

1.3 **CALCULATE** the mathematical mean variance of a given set of data.

1.4 Given the data, **CALCULATE** the probability of an event.

TERMINAL OBJECTIVE

2.0 **SOLVE** for problems involving the use of complex numbers.

ENABLING OBJECTIVES

2.1 **STATE** the definition of an imaginary number.

2.2 **STATE** the definition of a complex number.

2.3 **APPLY** the arithmetic operations of addition, subtraction, multiplication, and division to complex numbers.

TERMINAL OBJECTIVE

3.0 **SOLVE** for the unknowns in a problem through the application of matrix mathematics.

ENABLING OBJECTIVES

3.1 **DETERMINE** the dimensions of a given matrix.

3.2 **SOLVE** a given set of equations using Cramer's Rule.

TERMINAL OBJECTIVE

4.0 **DESCRIBE** the use of differentials and integration in mathematical problems.

ENABLING OBJECTIVES

4.1 **STATE** the graphical definition of a derivative.

4.2 **STATE** the graphical definition of an integral.

STATISTICS

This chapter will cover the basic concepts of statistics.

EO 1.1 STATE the definition of the following statistical terms:
a. Mean
b. Variance
c. Mean variance

EO 1.2 CALCULATE the mathematical mean of a given set of data.

EO 1.3 CALCULATE the mathematical mean variance of a given set of data.

EO 1.4 Given the data, CALCULATE the probability of an event.

In almost every aspect of an operator's work, there is a necessity for making decisions resulting in some significant action. Many of these decisions are made through past experience with other similar situations. One might say the operator has developed a method of intuitive inference: unconsciously exercising some principles of probability in conjunction with statistical inference following from observation, and arriving at decisions which have a high chance of resulting in expected outcomes. In other words, statistics is a method or technique which will enable us to approach a problem of determining a course of action in a systematic manner in order to reach the desired results.

Mathematically, statistics is the collection of great masses of numerical information that is summarized and then analyzed for the purpose of making decisions; that is, the use of past information is used to predict future actions. In this chapter, we will look at some of the basic concepts and principles of statistics.

Frequency Distribution

When groups of numbers are organized, or ordered by some method, and put into tabular or graphic form, the result will show the "frequency distribution" of the data.

Example:

A test was given and the following grades were received: the number of students receiving each grade is given in parentheses.

99(1), 98(2), 96(4), 92(7), 90(5), 88(13), 86(11), 83(7), 80(5), 78(4), 75(3), 60(1)

The data, as presented, is arranged in descending order and is referred to as an ordered array. But, as given, it is difficult to determine any trend or other information from the data. However, if the data is tabled and/or plotted some additional information may be obtained. When the data is ordered as shown, a frequency distribution can be seen that was not apparent in the previous list of grades.

Grades	Number of Occurrences	Frequency Distribution
99	1	1
98	11	2
96	1111	4
92	11111 11	7
90	11111	5
88	11111 11111 111	13
86	11111 11111 1	11
83	11111 11	7
80	11111	5
78	1111	4
75	111	3
	1	1

In summary, one method of obtaining additional information from a set of data is to determine the frequency distribution of the data. The frequency distribution of any one data point is the number of times that value occurs in a set of data. As will be shown later in this chapter, this will help simplify the calculation of other statistically useful numbers from a given set of data.

The Mean

One of the most common uses of statistics is the determination of the mean value of a set of measurements. The term "Mean" is the statistical word used to state the "average" value of a set of data. The mean is mathematically determined in the same way as the "average" of a group of numbers is determined.

The arithmetic mean of a set of N measurements, X_l, X_2, X_3, ..., X_N is equal to the sum of the measurements divided by the number of data points, N. Mathematically, this is expressed by the following equation:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where

$\bar{x}$	=	the mean
n	=	the number of values (data)
x_1	=	the first data point, x_2 = the second data point,....x_i = the ith data point
x_i	=	the ith data point, x_1 = the first data point, x_2 = the second data point, etc.

The symbol Sigma ($\sum$) is used to indicate summation, and i = 1 to n indicates that the values of x_i from $i = 1$ to $i = n$ are added. The sum is then divided by the number of terms added, n.

Example:

Determine the mean of the following numbers:

5, 7, 1, 3, 4

Solution:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{5}\sum_{i=1}^{5} x_i$$

where

$\bar{x}$ = the mean
n = the number of values (data) = 5
x_1 = 5, x_2 = 7, x_3 = 1, x_4 = 3, x_5 = 4

substituting

$\bar{x}$ = (5 + 7 + 1 + 3 + 4)/5 = 20/5 = 4

4 is the mean.

Example:

Find the mean of 67, 88, 91, 83, 79, 81, 69, and 74.

Solution:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The sum of the scores is 632 and n = 8, therefore

$$\bar{x} = \frac{632}{8}$$

$$\bar{x} = 79$$

In many cases involving statistical analysis, literally hundreds or thousands of data points are involved. In such large groups of data, the frequency distribution can be plotted and the calculation of the mean can be simplified by multiplying each data point by its frequency distribution, rather than by summing each value. This is especially true when the number of discrete values is small, but the number of data points is large.

Therefore, in cases where there is a recurring number of data points, like taking the mean of a set of temperature readings, it is easier to multiply each reading by its frequency of occurrence (frequency of distribution), then adding each of the multiple terms to find the mean. This is one application using the frequency distribution values of a given set of data.

Example:

Given the following temperature readings,

573, 573, 574, 574, 574, 574, 575, 575, 575, 575, 575, 576, 576, 576, 578

Solution:

Determine the frequency of each reading.

Frequency Distribution

Temperatures	Frequency (f)	(f)(x_i)
573	2	1146
574	4	2296
575	5	2875
576	3	1728
578	1	578
	15	8623

Then calculate the mean,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\bar{x} = \frac{2(573) + 4(574) + 5(575) + 3(576) + 1(578)}{15}$$

$$\bar{x} = \frac{8623}{15}$$

$$\bar{x} = 574.9$$

Variability

We have discussed the averages and the means of sets of values. While the mean is a useful tool in describing a characteristic of a set of numbers, sometimes it is valuable to obtain information about the mean. There is a second number that indicates how representative the mean is of the data. For example, in the group of numbers, 100, 5, 20, 2, the mean is 31.75. If these data points represent tank levels for four days, the use of the mean level, 31.75, to make a decision using tank usage could be misleading because none of the data points was close to the mean.

This spread, or distance, of each data point from the mean is called the *variance*. The variance of each data point is calculated by:

$$\text{Variance} = \bar{x} - x_i$$

where

x_i = each data point
$\bar{x}$ = mean

The variance of each data point does not provide us with any useful information. But if the mean of the variances is calculated, a very useful number is determined. The *mean variance* is the average value of the variances of a set of data. The mean variance is calculated as follows:

$$\text{Mean Variance} = \frac{1}{n}\sum_{i=1}^{n} \left| x_i - \bar{x} \right|$$

The mean variance, or mean deviation, can be calculated and used to make judgments by providing information on the quality of the data. For example, if you were trying to decide whether to buy stock, and all you knew was that this month's average price was $10, and today's price is $9, you might be tempted to buy some. But, if you also knew that the mean variance in the stock's price over the month was $6, you would realize the stock had fluctuated widely during the month. Therefore, the stock represented a more risky purchase than just the average price indicated.

It can be seen that to make sound decisions using statistical data, it is important to analyze the data thoroughly before making any decisions.

Example:

Calculate the variance and mean variance of the following set of hourly tank levels. Assume the tank is a 100 gal. tank. Based on the mean and the mean variance, would you expect the tank to be able to accept a 40% (40 gal.) increase in level at any time?

1:00 - 40%	6:00 - 38%	11:00- 34%
2:00 - 38%	7:00 - 34%	12:00- 30%
3:00 - 28%	8:00 - 28%	1:00 - 40%
4:00 - 28%	9:00 - 40%	2:00 - 36%
5:00 - 40%	10:00- 38%	

Solution:

The mean is

$$[40(4)+38(3)+36+34(2)+30+28(3)]/14 = 492/14 = 35.1$$

The mean variance is:

$$\frac{1}{14}\left(\left|40 - 35.1\right| + \left|38 - 35.1\right| + \left|28 - 35.1\right| + \ldots\left|36 - 35.1\right|\right) =$$

$$\frac{1}{14}\,(57.8) = 4.12$$

From the tank mean of 35.1%, it can be seen that a 40% increase in level will statistically fit into the tank; 35.1 + 40 <100%. But, the mean doesn't tell us if the level varies significantly over time. Knowing the mean variance is 4.12% provides the additional information. Knowing the mean variance also allows us to infer that the level at any given time (most likely) will not be greater than 35.1 + 4.12 = 39.1%; and 39.1 + 40 is still less than 100%. Therefore, it is a good assumption that, in the near future, a 40% level increase will be accepted by the tank without any spillage.

Normal Distribution

The concept of a normal distribution curve is used frequently in statistics. In essence, a normal distribution curve results when a large number of random variables are observed in nature, and their values are plotted. While this "distribution" of values may take a variety of shapes, it is interesting to note that a very large number of occurrences observed in nature possess a frequency distribution which is approximately bell-shaped, or in the form of a normal distribution, as indicated in Figure 1.

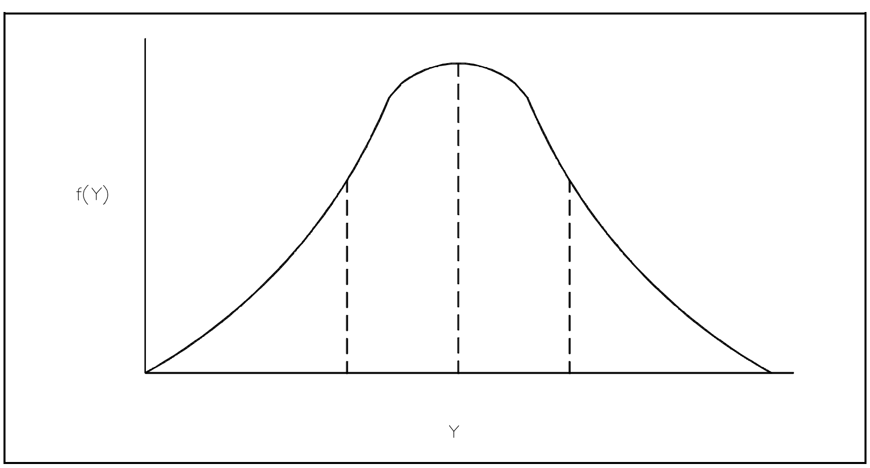

Figure 1 Graph of a Normal Probability Distribution

The significance of a normal distribution existing in a series of measurements is two fold. First, it explains why such measurements tend to possess a normal distribution; and second, it provides a valid basis for statistical inference. Many estimators and decision makers that are used to make inferences about large numbers of data, are really sums or averages of those measurements. When these measurements are taken, especially if a large number of them exist, confidence can be gained in the values, if these values form a bell-shaped curve when plotted on a distribution basis.

Probability

If E_1 is the number of heads, and E_2 is the number of tails, $E_1/(E_1 + E_2)$ is an experimental determination of the probability of heads resulting when a coin is flipped.

$$P(E_l) = n/N$$

By definition, the probability of an event must be greater than or equal to 0, and less than or equal to l. In addition, the sum of the probabilities of all outcomes over the entire "event" must add to equal l. For example, the probability of heads in a flip of a coin is 50%, the probability of tails is 50%. If we assume these are the only two possible outcomes, 50% + 50%, the two outcomes, equals 100%, or 1.

The concept of probability is used in statistics when considering the reliability of the data or the measuring device, or in the correctness of a decision. To have confidence in the values measured or decisions made, one must have an assurance that the probability is high of the measurement being true, or the decision being correct.

To calculate the probability of an event, the number of successes (s), and failures (f), must be determined. Once this is determined, the probability of the success can be calculated by:

$$p = \frac{s}{s + f}$$

where

s + f = n = number of tries.

Example:

Using a die, what is the probability of rolling a three on the first try?

Solution:

First, determine the number of possible outcomes. In this case, there are 6 possible outcomes. From the stated problem, the roll is a success only if a 3 is rolled. There is only 1 success outcome and 5 failures. Therefore,

$$\text{Probability} = 1/(1+5) = 1/6$$

In calculating probability, the probability of a series of independent events equals the product of probability of the individual events.

Example:

Using a die, what is the probability of rolling two 3s in a row?

Solution:

From the previous example, there is a 1/6 chance of rolling a three on a single throw. Therefore, the chance of rolling two threes is:

$$1/6 \times 1/6 = 1/36$$

one in 36 tries.

Example:

An elementary game is played by rolling a die and drawing a ball from a bag containing 3 white and 7 black balls. The player wins whenever he rolls a number less than 4 and draws a black ball. What is the probability of winning in the first attempt?

Solution:

There are 3 successful outcomes for rolling less than a 4, (i.e. 1,2,3). The probability of rolling a 3 or less is:

$$3/(3+3) = 3/6 = 1/2 \text{ or } 50\%.$$

The probability of drawing a black ball is:

7/(7+3) = 7/10.

Therefore, the probability of both events happening at the same time is:

7/10 x 1/2 = 7/20.

Summary

The important information in this chapter is summarized below.

Statistics Summary

Mean	-	The sum of a group of values divided by the number of values.
Frequency Distribution	-	An arrangement of statistical data that exhibits the frequency of the occurrence of the values of a variable.
Variance	-	The difference of a data point from the mean.
Mean Variance	-	The mean or average of the absolute values of each data point's variance.
Probability of Success	-	The chances of being successful out of a number of tries.

$$P = \frac{s}{s+f}$$

IMAGINARY AND COMPLEX NUMBERS

This chapter will cover the definitions and rules for the application of imaginary and complex numbers.

EO 2.1 **STATE the definition of an imaginary number.**

EO 2.2 **STATE the definition of a complex number.**

EO 2.3 **APPLY the arithmetic operations of addition, subtraction, and multiplication, and division to complex numbers.**

Imaginary and complex numbers are entirely different from any kind of number used up to this point. These numbers are generated when solving some quadratic and higher degree equations. Imaginary and complex numbers become important in the study of electricity; especially in the study of alternating current circuits.

Imaginary Numbers

Imaginary numbers result when a mathematical operation yields the square root of a negative number. For example, in solving the quadratic equation $x^2 + 25 = 0$, the solution yields $x^2 = -25$. Thus, the roots of the equation are $x = \pm\sqrt{-25}$. The square root of (-25) is called an imaginary number. Actually, any even root (i.e. square root, 4th root, 6th root, etc.) of a negative number is called an imaginary number. All other numbers are called real numbers. The name "imaginary" may be somewhat misleading since imaginary numbers actually exist and can be used in mathematical operations. They can be added, subtracted, multiplied, and divided.

Imaginary numbers are written in a form different from real numbers. Since they are radicals, they can be simplified by factoring. Thus, the imaginary number $\sqrt{-25}$ equals $\sqrt{(25)(-1)}$, which equals $5\sqrt{-1}$. Similarly, $\sqrt{-9}$ equals $\sqrt{(9)(-1)}$, which equals $3\sqrt{-1}$. All imaginary numbers can be simplified in this way. They can be written as the product of a real number and $\sqrt{-1}$. In order to further simplify writing imaginary numbers, the imaginary unit i is defined as $\sqrt{-1}$. Thus, the imaginary number, $\sqrt{-25}$, which equals $5\sqrt{-1}$, is written as $5i$, and the imaginary number, $\sqrt{-9}$, which equals $3\sqrt{-1}$, is written $3i$. In using imaginary numbers in electricity, the imaginary unit is often represented by j, instead of i, since i is the common notation for electrical current.

Imaginary numbers are added or subtracted by writing them using the imaginary unit *i* and then adding or subtracting the real number coefficients of *i*. They are added or subtracted like algebraic terms in which the imaginary unit *i* is treated like a literal number. Thus, $\sqrt{-25}$ and $\sqrt{-9}$ are added by writing them as 5*i* and 3*i* and adding them like algebraic terms. The result is 8*i* which equals $8\sqrt{-1}$ or $\sqrt{-64}$. Similarly, $\sqrt{-9}$ subtracted from $\sqrt{-25}$ equals 3*i* subtracted from 5*i* which equals 2*i* or $2\sqrt{-1}$ or $\sqrt{-4}$.

Example:

Combine the following imaginary numbers:

Solution:

$$\sqrt{-16} + \sqrt{-36} - \sqrt{-49} - \sqrt{-1} =$$

$$\begin{aligned}\sqrt{-16} + \sqrt{-36} - \sqrt{-49} - \sqrt{-1} &= 4i + 6i - 7i - i \\ &= 10i - 8i \\ &= 2i\end{aligned}$$

Thus, the result is $2i = 2\sqrt{-1} = \sqrt{-4}$

Imaginary numbers are multiplied or divided by writing them using the imaginary unit *i*, and then multiplying or dividing them like algebraic terms. However, there are several basic relationships which must also be used to multiply or divide imaginary numbers.

$$\begin{aligned} i^2 &= (i)(i) = (\sqrt{-1})(\sqrt{-1}) = -1 \\ i^3 &= (i^2)(i) = (-1)(i) = -i \\ i^4 &= (i^2)(i^2) = (-1)(-1) = +1 \end{aligned}$$

Using these basic relationships, for example, $(\sqrt{-25})(\sqrt{-4})$ equals $(5i)(2i)$ which equals $10i^2$. But, i^2 equals -1. Thus, $10i^2$ equals (10)(-1) which equals -10.

Any square root has two roots, i.e., a statement $x^2 = 25$ is a quadratic and has roots

$x = \pm 5$ since $+5^2 = 25$ and $(-5) \times (-5) = 25$.

Similarly,

$\sqrt{-25} = \pm 5i$
$\sqrt{-4} = \pm 2i$
and
$\sqrt{-25}\ \sqrt{-4} = \pm 10$.

Example 1:

Multiply $\sqrt{-2}$ and $\sqrt{-32}$.

Solution:

$$(\sqrt{-2})\,(\sqrt{-32}) = (\sqrt{2}\,i)\,(\sqrt{32}\,i)$$

$$= \sqrt{(2)\,(32)}\ i^2$$

$$= \sqrt{64}\ (-1)$$

$$= 8\ (-1)$$

$$= -8$$

Example 2:

Divide $\sqrt{-48}$ by $\sqrt{-3}$.

Solution:

$$\frac{\sqrt{-48}}{\sqrt{-3}} = \frac{\sqrt{48}i}{\sqrt{3}i}$$

$$= \sqrt{\frac{48}{3}}$$

$$= \sqrt{16}$$

$$= 4$$

Complex Numbers

Complex numbers are numbers which consist of a real part and an imaginary part. The solution of some quadratic and higher degree equations results in complex numbers. For example, the roots of the quadratic equation, $x^2 - 4x + 13 = 0$, are complex numbers. Using the quadratic formula yields two complex numbers as roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$x = \frac{4 \pm \sqrt{-36}}{2}$$

$$x = \frac{4 \pm 6i}{2}$$

$$x = 2 \pm 3i$$

The two roots are $2 + 3i$ and $2 - 3i$; they are both complex numbers. 2 is the real part; $+3i$ and $-3i$ are the imaginary parts. The general form of a complex number is $a + bi$, in which "a" represents the real part and "bi" represents the imaginary part.

Complex numbers are added, subtracted, multiplied, and divided like algebraic binomials. Thus, the sum of the two complex numbers, $7 + 5i$ and $2 + 3i$ is $9 + 8i$, and $7 + 5i$ minus $2 + 3i$, is $5 + 2i$. Similarly, the product of $7 + 5i$ and $2 + 3i$ is $14 + 31i + 15i^2$. But i^2 equals -1. Thus, the product is $14 + 31i + 15(-1)$ which equals $-1 + 31i$.

Example 1:

Combine the following complex numbers:

$$(4 + 3i) + (8 - 2i) - (7 + 3i) =$$

Solution:

$$\begin{aligned}(4 + 3i) + (8 - 2i) - (7 + 3i) &= (4 + 8 - 7) + (3 - 2 - 3)i \\ &= 5 - 2i\end{aligned}$$

Example 2:

Multiply the following complex numbers:

$(3 + 5i)(6 - 2i) =$

Solution:

$$\begin{aligned}(3 + 5i)(6 - 2i) &= 18 + 30i - 6i - 10i^2 \\ &= 18 + 24i - 10(-1) \\ &= 28 + 24i\end{aligned}$$

Example 3:

Divide $(6+8i)$ by 2.

Solution:

$$\frac{6 + 8i}{2} = \frac{6}{2} + \frac{8}{2}i = 3 + 4i$$

A difficulty occurs when dividing one complex number by another complex number. To get around this difficulty, one must eliminate the imaginary portion of the complex number from the denominator, when the division is written as a fraction. This is accomplished by multiplying the numerator and denominator by the *conjugate* form of the denominator. The *conjugate* of a complex number is that complex number written with the opposite sign for the imaginary part. For example, the conjugate of $4+5i$ is $4-5i$.

This method is best demonstrated by example.

Example: $(4 + 8i) \div (2 - 4i)$

Solution:

$$\begin{aligned}\frac{4 + 8i}{2 - 4i} \cdot \frac{2 + 4i}{2 + 4i} &= \frac{8 + 32i + 32i^2}{4 - 16i^2} \\ &= \frac{8 + 32i + 32(-1)}{4-16(-1)} \\ &= \frac{-24 + 32i}{20} \\ &= -\frac{6}{5} + \frac{8}{5}i\end{aligned}$$

Summary

The important information from this chapter is summartized below.

Imaginary and Complex Numbers Summary

Imaginary Number

- An Imaginary number is the square root of a negative number.

Complex Number

- A complex number is any number that contains both a real and imaginary term.

Addition and Subtraction of Complex Numbers

- Add/subtract the real terms together, and add/subtract the imaginary terms of each complex number together. The result will be a complex number.

Multiplication of Complex Numbers

- Treat each complex number as an algebraic term and multiply/divide using rules of algebra. The result will be a complex number.

Division of Complex Numbers

- Put division in fraction form and multiply numerator and denominator by the conjugate of the denominator.

Rules of the Imaginary Number *i*

- $i^2 = (i)(i) = -1$
  $i^3 = (i^2)(i) = (-1)(i) = -i$
  $i^4 = (i^2)(i^2) = (-1)(-1) = +1$

MATRICES AND DETERMINANTS

This chapter will explain the idea of matrices and determinate and the rules needed to apply matrices in the solution of simultaneous equations.

EO 3.1 DETERMINE the dimensions of a given matrix.

EO 3.2 SOLVE a given set of equations using Cramer's Rule.

In the real world, many times the solution to a problem containing a large number of variables is required. In both physics and electrical circuit theory, many problems will be encountered which contain multiple simultaneous equations with multiple unknowns. These equations can be solved using the standard approach of eliminating the variables or by one of the other methods. This can be difficult and time-consuming. To avoid this problem, and easily solve families of equations containing multiple unknowns, a type of math was developed called Matrix theory.

Once the terminology and basic manipulations of matrices are understood, matrices can be used to readily solve large complex systems of equations.

The Matrix

We define a matrix as any rectangular array of numbers. Examples of matrices may be formed from the coefficients and constants of a system of linear equations: that is,

$$2x - 4y = 7$$
$$3x + y = 16$$

can be written as follows.

$$\begin{bmatrix} 2 & -4 & 7 \\ 3 & 1 & 16 \end{bmatrix}$$

The numbers used in the matrix are called elements. In the example given, we have three columns and two rows of elements. The number of rows and columns are used to determine the dimensions of the matrix. In our example, the dimensions of the matrix are 2 x 3, having 2 rows and 3 columns of elements. In general, the dimensions of a matrix which have *m* rows and *n* columns is called an *m* x *n* matrix.

A matrix with only a single row or a single column is called either a row or a column matrix. A matrix which has the same number of rows as columns is called a square matrix. Examples of matrices and their dimensions are as follows:

$$\begin{bmatrix} 1 & 7 & 6 \\ 2 & 4 & 8 \end{bmatrix} = 2 \text{ x } 3$$

$$\begin{bmatrix} 1 & 7 \\ 6 & 2 \\ 3 & 5 \end{bmatrix} = 3 \text{ x } 2$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 3 \text{ x } 1$$

We will use capital letters to describe matrices. We will also include subscripts to give the dimensions.

$$A_{2 \times 3} = \begin{bmatrix} 1 & 3 & 3 \\ 5 & 6 & 7 \end{bmatrix}$$

Two matrices are said to be equal if, and only if, they have the same dimensions, and their corresponding elements are equal. The following are all equal matrices:

$$\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{1} \\ \frac{6}{3} & 4 \end{bmatrix}$$

Addition of Matrices

Matrices may only be added if they both have the same dimensions. To add two matrices, each element is added to its corresponding element. The sum matrix has the same dimensions as the two being added.

Example:

Add matrix A to matrix B.

$$A = \begin{bmatrix} 6 & 2 & 6 \\ -1 & 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 6+2 & 2+1 & 6+3 \\ -1+0 & 3-3 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 9 \\ -1 & 0 & 6 \end{bmatrix}$$

Multiplication of a Scalar and a Matrix

When multiplying a matrix by a scalar (or number), we write "scalar K times matrix A." Each element of the matrix is multiplied by the scalar. By example:

$$K = 3 \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

then

$$3 \text{ x } A = 3\begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3 & 3 \cdot 3 \\ 1 \cdot 3 & 7 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ 3 & 21 \end{bmatrix}$$

Multiplication of a Matrix by a Matrix

To multiply two matrices, the first matrix must have the same number of rows (m) as the second matrix has columns (n). In other words, m of the first matrix must equal n of the second matrix. For example, a 2 x 1 matrix can be multiplied by a 1 x 2 matrix,

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} ax & bx \\ ay & by \end{bmatrix}$$

or a 2 x 2 matrix can be multiplied by a 2 x 2. If an m x n matrix is multiplied by an n x p matrix, then the resulting matrix is an m x p matrix. For example, if a 2 x 1 and a 1 x 2 are multiplied, the result will be a 2 x 2. If a 2 x 2 and a 2 x 2 are multiplied, the result will be a 2 x 2.

To multiply two matrices, the following pattern is used:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$C = A \cdot B = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}$$

In general terms, a matrix C which is a product of two matrices, A and B, will have elements given by the following.

$$c_{\text{ij}} = a_{\text{i1}}b_{\text{1j}} + a_{\text{j2}}b_{\text{2j}} + + + \ldots + a_{\text{in}}b_{\text{nj}}$$

where

i = ith row
j = jth column

Example:

Multiply the matrices A x B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

Solution:

$$A \cdot B = \begin{bmatrix} (1x3)+(2x0) & (1x5)+(2x6) \\ (3x3)+(4x0) & (3x5)+(4x6) \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & 5+12 \\ 9+0 & 15+24 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 17 \\ 9 & 39 \end{bmatrix}$$

It should be noted that the multiplication of matrices is not usually commutative.

The Determinant

Square matrixes have a property called a determinant. When a determinant of a matrix is written, it is symbolized by vertical bars rather than brackets around the numbers. This differentiates the determinant from a matrix. The determinant of a matrix is the reduction of the matrix to a single scalar number. The determinant of a matrix is found by "expanding" the matrix. There are several methods of "expanding" a matrix and calculating it's determinant. In this lesson, we will only look at a method called "expansion by minors."

Before a large matrix determinant can be calculated, we must learn how to calculate the determinant of a 2 x 2 matrix. By definition, the determinant of a 2 x 2 matrix is calculated as follows:

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: Find the determinant of A.

$$A = \begin{vmatrix} 6 & 2 \\ -1 & 3 \end{vmatrix}$$

Solution:

$$\begin{aligned} A &= (6 \cdot 3) - (-1 \cdot 2) \\ &= 18 - (-2) \\ &= 18 + 2 \\ &= 20 \end{aligned}$$

To expand a matrix larger than a 2 x 2 requires that it be simplified down to several 2 x 2 matrices, which can then be solved for their determinant. It is easiest to explain the process by example.

Given the 3 x 3 matrix:

$$\begin{matrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{matrix}$$

Any single row or column is picked. In this example, column one is selected. The matrix will be expanded using the elements from the first column. Each of the elements in the selected column will be multiplied by its minor starting with the first element in the column (1). A line is then drawn through all the elements in the same row and column as 1. Since this is a 3 x 3 matrix, that leaves a minor or 2 x 2 determinant. This resulting 2 x 2 determinant is called the minor of the element in the first row first column.

$$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{bmatrix}$$

$$1 \begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix}$$

The minor of element 4 is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{bmatrix}$$

$$4 \begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

The minor of element 5 is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{bmatrix}$$

$$5 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

Each element is given a sign based on its position in the original determinant.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The sign is positive (negative) if the sum of the row plus the column for the element is even (odd). This pattern can be expanded or reduced to any size determinant. The positive and negative signs are just alternated.

Each minor is now multiplied by its signed element and the determinant of the resulting 2 x 2 calculated.

$$1\begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix} = 1\left[(1 \cdot 3) - (2 \cdot 6)\right] = 3 - (12) = -9$$

$$-4\begin{bmatrix} 3 & 1 \\ 6 & 3 \end{bmatrix} = -4\left[(3 \cdot 3) - (1 \cdot 6)\right] = -4\left[9 - 6\right] = -12$$

$$5\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = 5\left[(3 \cdot 2) - (1 \cdot 1)\right] = 5\left[6 - 1\right] = 25$$

Determinant = (-9) + (-12) + 25 = 4

Example:

Find the determinant of the following 3 x 3 matrix, expanding about row 1.

$$\begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{vmatrix}$$

Solution:

First Minor

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{bmatrix} = 3\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix} = 3\,(20 - 6) = 3\,(14) = 42$$

Second Minor

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{bmatrix} = -1\begin{vmatrix} 4 & 6 \\ 0 & 4 \end{vmatrix} = -1\,(16 - 0) = -1\,(16) = -16$$

Third Minor

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{bmatrix} = 3\begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} = 3\,(4 - 0) = 3\,(4) = 12$$

$$\text{Determinant} = 42 + (-16) + (12) = 38$$

Using Matrices to Solve System of Linear Equation (Cramer's Rule)

Matrices and their determinant can be used to solve a system of equations. This method becomes especially attractive when large numbers of unknowns are involved. But the method is still useful in solving algebraic equations containing two and three unknowns.

In part one of this chapter, it was shown that equations could be organized such that their coefficients could be written as a matrix.

$$ax + by = c$$
$$ex + fy = g$$

where:

x and y are variables
a, b, e, and f are the coefficients
c and g are constants

The equations can be rewritten in matrix form as follows:

$$\begin{bmatrix} a & b \\ e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ g \end{bmatrix}$$

To solve for each variable, the matrix containing the constants (*c*,*g*) is substituted in place of the column containing the coefficients of the variable that we want to solve for (*a*,*e* or *b*,*f*). This new matrix is divided by the original coefficient matrix. This process is call "Cramer's Rule."

Example:

In the above problem to solve for *x*,

$$x = \frac{\begin{vmatrix} c & b \\ g & f \end{vmatrix}}{\begin{vmatrix} c & b \\ g & f \end{vmatrix}}$$

and to solve for *y*,

$$y = \frac{\begin{vmatrix} a & c \\ e & g \end{vmatrix}}{\begin{vmatrix} a & b \\ e & f \end{vmatrix}}$$

Example:

Solve the following two equations:

$$x + 2y = 4$$
$$-x + 3y = 1$$

Solution:

$$x = \frac{\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}}$$

solving each 2 x 2 for its determinant,

$$x = \frac{[(4\cdot3) - (1\cdot2)]}{[(1\cdot3) - (-1\cdot2)]} = \frac{12 - 2}{3 + 2} = \frac{10}{5} = 2$$

$$y = \frac{[(1\cdot1) - (-1\cdot4)]}{[(1\cdot3) - (-1\cdot2)]} = \frac{1 + 4}{3 + 2} = \frac{5}{5} = 1$$

$$x = 2 \quad \text{and} \quad y = 1$$

A 3 x 3 is solved by using the same logic, except each 3 x 3 must be expanded by minors to solve for the determinant.

Example:

Given the following three equations, solve for the three unknowns.

$2x + 3y - z = 2$
$x - 2y + 2z = -10$
$3x + y - 2z = 1$

Solution:

$$x = \frac{\begin{vmatrix} 2 & 3 & -1 \\ -10 & -2 & 2 \\ 1 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} 2 & 2 & -1 \\ 1 & -10 & 2 \\ 3 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} 2 & 3 & 2 \\ 1 & -2 & -10 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

Expanding the top matrix for *x* using the elements in the bottom row gives:

$$1 \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 & -1 \\ -10 & 2 \end{bmatrix} + (-2) \begin{bmatrix} 2 & 3 \\ -10 & -2 \end{bmatrix} =$$

$$1\,(6 - 2) + (-1)\,(4 - 10) + (-2)\,(-4 + 30) =$$

$$4 + 6 - 52 = -42$$

Expanding the bottom matrix for *x* using the elements in the first column gives:

$$2\begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix} + (-1)\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} + 3\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} =$$

$$2(4 - 2) + (-1)(-6 + 1) + 3(6 - 2) =$$

$$4 + 5 + 12 = 21$$

This gives:

$$x = \frac{-42}{21} = -2$$

y and *z* can be expanded using the same method.

$$y = 1$$
$$z = -3$$

Summary

The use of matrices and determinants is summarized below.

Matrices and Determinant Summary

The dimensions of a matrix are given as *m* x *n*, where *m* = number of rows and *n* = number of columns.

The use of determinants and matrices to solve linear equations is done by:

- placing the coefficients and constants into a determinant format.
- substituting the constants in place of the coefficients of the variable to be solved for.
- dividing the new-formed substituted determinant by the original determinant of coefficients.
- expanding the determinant.

CALCULUS

Many practical problems can be solved using arithmetic and algebra; however, many other practical problems involve quantities that cannot be adequately described using numbers which have fixed values.

EO 4.1 STATE the graphical definition of a derivative.

EO 4.2 STATE the graphical definition of an integral.

Dynamic Systems

Arithmetic involves numbers that have fixed values. Algebra involves both literal and arithmetic numbers. Although the literal numbers in algebraic problems can change value from one calculation to the next, they also have fixed values in a given calculation. When a weight is dropped and allowed to fall freely, its velocity changes continually. The electric current in an alternating current circuit changes continually. Both of these quantities have a different value at successive instants of time. Physical systems that involve quantities that change continually are called dynamic systems. The solution of problems involving dynamic systems often involves mathematical techniques different from those described in arithmetic and algebra. Calculus involves all the same mathematical techniques involved in arithmetic and algebra, such as addition, subtraction, multiplication, division, equations, and functions, but it also involves several other techniques. These techniques are not difficult to understand because they can be developed using familiar physical systems, but they do involve new ideas and terminology.

There are many dynamic systems encountered in nuclear facility work. The decay of radioactive materials, the startup of a reactor, and a power change on a turbine generator all involve quantities which change continually. An analysis of these dynamic systems involves calculus. Although the operation of a nuclear facility does not require a detailed understanding of calculus, it is most helpful to understand certain of the basic ideas and terminology involved. These ideas and terminology are encountered frequently, and a brief introduction to the basic ideas and terminology of the mathematics of dynamic systems is discussed in this chapter.

Differentials and Derivatives

One of the most commonly encountered applications of the mathematics of dynamic systems involves the relationship between position and time for a moving object. Figure 2 represents an object moving in a straight line from position P_1 to position P_2. The distance to P_1 from a fixed reference point, point 0, along the line of travel is represented by S_1; the distance to P_2 from point 0 by S_2.

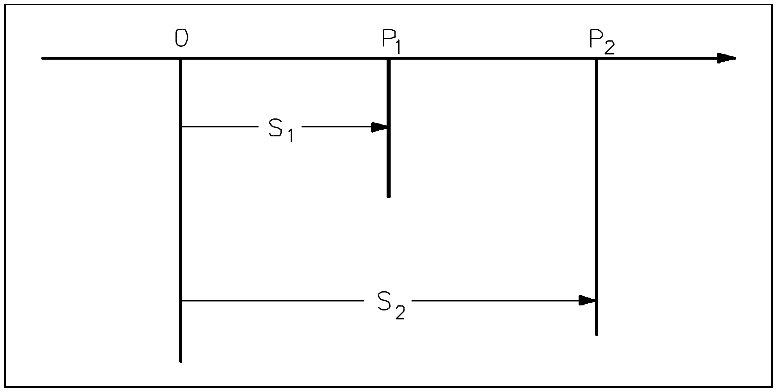

Figure 2 Motion Between Two Points

If the time recorded by a clock, when the object is at position P_1 is t_1, and if the time when the object is at position P_2 is t_2, then the average velocity of the object between points P_1 and P_2 equals the distance traveled, divided by the elapsed time.

$$V_{av} = \frac{S_2 - S_1}{t_2 - t_1} \tag{5-1}$$

If positions P_1 and P_2 are close together, the distance traveled and the elapsed time are small. The symbol Δ, the Greek letter delta, is used to denote changes in quantities. Thus, the average velocity when positions P_1 and P_2 are close together is often written using deltas.

$$V_{av} = \frac{\Delta S}{\Delta t} = \frac{S_2 - S_1}{t_2 - t_1} \tag{5-2}$$

Although the average velocity is often an important quantity, in many cases it is necessary to know the velocity at a given instant of time. This velocity, called the instantaneous velocity, is not the same as the average velocity, unless the velocity is not changing with time.

Using the graph of displacement, S, versus time, t, in Figure 3, we will try to describe the concept of the derivative.

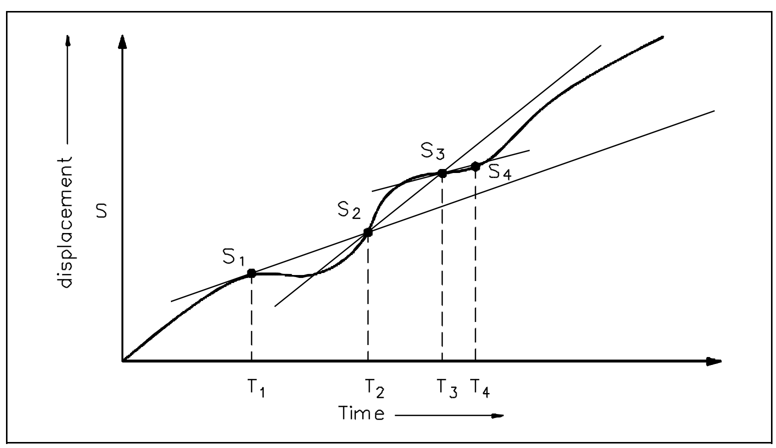

Figure 3 Displacement Versus Time

Using equation 5-1 we find the average velocity from S_1 to S_2 is $\frac{S_2 - S_1}{t_2 - t_1}$. If we connect the points S_1 and S_2 by a straight line we see it does not accurately reflect the slope of the curved line through all the points between S_1 and S_2. Similarly, if we look at the average velocity between time t_2 and t_3 (a smaller period of time), we see the straight line connecting S_2 and S_3 more closely follows the curved line. Assuming the time between t_3 and t_4 is less than between t_2 and t_3, the straight line connecting S_3 and S_4 very closely approximates the curved line between S_3 and S_4.

As we further decrease the time interval between successive points, the expression $\frac{\Delta S}{\Delta t}$ more closely approximates the slope of the displacement curve. As $\Delta t \rightarrow 0$, $\frac{\Delta S}{\Delta t}$ approaches the instantaneous velocity. The expression for the derivative (in this case the slope of the displacement curve) can be written $\frac{dS}{dt} = \lim_{\Delta t \rightarrow o} \frac{\Delta S}{\Delta t}$. In words, this expression would be "the derivative of S with respect to time (t) is the limit of $\frac{\Delta S}{\Delta t}$ as Δt approaches *0*."

$$V = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} \qquad (5\text{-}3)$$

The symbols *ds* and *dt* are not products of *d* and *s*, or of *d* and *t*, as in algebra. Each represents a single quantity. They are pronounced "dee-ess" and "dee-tee," respectively. These expressions and the quantities they represent are called differentials. Thus, *ds* is the differential of *s* and *dt* is the differential of *t*. These expressions represent incremental changes, where *ds* represents an incremental change in distance *s*, and *dt* represents an incremental change in time *t*.

The combined expression *ds/dt* is called a derivative; it is the derivative of *s* with respect to *t*. It is read as "dee-ess dee-tee." *dz/dx* is the derivative of *z* with respect to *x*; it is read as "dee-zee dee-ex." In simplest terms, a derivative expresses the rate of change of one quantity with respect to another. Thus, *ds/dt* is the rate of change of distance with respect to time. Referring to figure 3, the derivative *ds/dt* is the instantaneous velocity at any chosen point along the curve. This value of instantaneous velocity is numerically equal to the slope of the curve at that chosen point.

While the equation for instantaneous velocity, $V = ds/dt$, may seem like a complicated expression, it is a familiar relationship. Instantaneous velocity is precisely the value given by the speedometer of a moving car. Thus, the speedometer gives the value of the rate of change of distance with respect to time; it gives the derivative of *s* with respect to *t*; i.e. it gives the value of *ds/dt*.

The ideas of differentials and derivatives are fundamental to the application of mathematics to dynamic systems. They are used not only to express relationships among distance traveled, elapsed time and velocity, but also to express relationships among many different physical quantities. One of the most important parts of understanding these ideas is having a physical interpretation of their meaning. For example, when a relationship is written using a differential or a derivative, the physical meaning in terms of incremental changes or rates of change should be readily understood.

When expressions are written using deltas, they can be understood in terms of changes. Thus, the expression ΔT, where T is the symbol for temperature, represents a change in temperature. As previously discussed, a lower case delta, d, is used to represent very small changes. Thus, dT represents a very small change in temperature. The fractional change in a physical quantity is the change divided by the value of the quantity. Thus, dT is an incremental change in temperature, and dT/T is a fractional change in temperature. When expressions are written as derivatives, they can be understood in terms of rates of change. Thus, dT/dt is the rate of change of temperature with respect to time.

Examples:

1. Interpret the expression $\Delta V/V$, and write it in terms of a differential. $\Delta V/V$ expresses the fractional change of velocity. It is the change in velocity divided by the velocity. It can be written as a differential when ΔV is taken as an incremental change.

   $$\frac{\Delta V}{V} \text{ may be written as } \frac{dV}{V}$$

2. Give the physical interpretation of the following equation relating the work W done when a force F moves a body through a distance x.

   $$dW = Fdx$$

   This equation includes the differentials dW and dx which can be interpreted in terms of incremental changes. The incremental amount of work done equals the force applied multiplied by the incremental distance moved.

3. Give the physical interpretation of the following equation relating the force, F, applied to an object, its mass m, its instantaneous velocity v and time t.

$$F = m\frac{dv}{dt}$$

This equation includes the derivative dv/dt; the derivative of the velocity with respect to time. It is the rate of change of velocity with respect to time. The force applied to an object equals the mass of the object multiplied by the rate of change of velocity with respect to time.

4. Give the physical interpretation of the following equation relating the acceleration a, the velocity v, and the time t.

$$a = \frac{dv}{dt}$$

This equation includes the derivative dv/dt; the derivative of the velocity with respect to time. It is a rate of change. The acceleration equals the rate of change of velocity with respect to time.

Graphical Understanding of Derivatives

A function expresses a relationship between two or more variables. For example, the distance traveled by a moving body is a function of the body's velocity and the elapsed time. When a functional relationship is presented in graphical form, an important understanding of the meaning of derivatives can be developed.

Figure 4 is a graph of the distance traveled by an object as a function of the elapsed time. The functional relationship shown is given by the following equation:

$$s = 40t \tag{5-4}$$

The instantaneous velocity v, which is the velocity at a given instant of time, equals the derivative of the distance traveled with respect to time, ds/dt. It is the rate of change of s with respect to t.

The value of the derivative *ds/dt* for the case plotted in Figure 4 can be understood by considering small changes in the two variables *s* and *t*.

$$\frac{\Delta s}{\Delta t} = \frac{(s + \Delta s) - s}{(t + \Delta t) - t}$$

The values of $(s + \Delta s)$ and s in terms of $(t + \Delta t)$ and t, using Equation 5-4 can now be substituted into this expression. At time t, $s = 40t$; at time $t + \Delta t$, $s + \Delta s = 40(t + \Delta t)$.

$$\frac{\Delta s}{\Delta t} = \frac{40(t + \Delta t) - 40t}{(t + \Delta t) - t}$$

$$\frac{\Delta s}{\Delta t} = \frac{40t + 40(\Delta t) - 40t}{t + \Delta t - t}$$

$$\frac{\Delta s}{\Delta t} = \frac{40(\Delta t)}{\Delta t}$$

$$\frac{\Delta s}{\Delta t} = 40$$

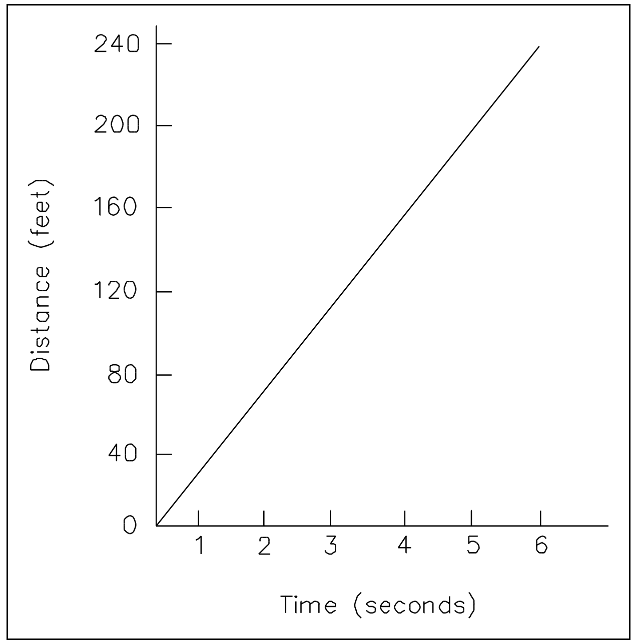

Figure 4 Graph of Distance vs. Time

The value of the derivative *ds/dt* in the case plotted in Figure 4 is a constant. It equals 40 ft/s. In the discussion of graphing, the slope of a straight line on a graph was defined as the change in y, Δy, divided by the change in x, Δx. The slope of the line in Figure 4 is $\Delta s/\Delta t$ which, in this case, is the value of the derivative *ds/dt*. Thus, derivatives of functions can be interpreted in terms of the slope of the graphical plot of the function. Since the velocity equals the derivative of the distance s with respect to time t, *ds/dt*, and since this derivative equals the slope of the plot of distance versus time, the velocity can be visualized as the slope of the graphical plot of distance versus time.

For the case shown in Figure 4, the velocity is constant. Figure 5 is another graph of the distance traveled by an object as a function of the elapsed time. In this case the velocity is not constant. The functional relationship shown is given by the following equation:

$$s = 10t^2 \tag{5-5}$$

The instantaneous velocity again equals the value of the derivative *ds/dt*. This value is changing with time. However, the instantaneous velocity at any specified time can be determined. First, small changes in *s* and *t* are considered.

$$\frac{\Delta s}{\Delta t} = \frac{(s + \Delta s) - s}{(t + \Delta t) - t}$$

The values of ($s + \Delta s$) and s in terms of ($t + \Delta t$) and t using Equation 5-5, can then be substituted into this expression. At time t, $s = 10t^2$; at time $t + \Delta t$, $s + \Delta s = 10(t + \Delta t)^2$. The value of $(t + \Delta t)^2$ equals $t^2 + 2t(\Delta t) + (\Delta t)^2$; however, for incremental values of Δt, the term $(\Delta t)^2$ is so small, it can be neglected. Thus, $(t + \Delta t)^2 = t^2 + 2t(\Delta t)$.

$$\frac{\Delta s}{\Delta t} = \frac{10[t^2 + 2t(\Delta t)] - 10t^2}{(t + \Delta t) - t}$$

$$\frac{\Delta s}{\Delta t} = \frac{10t^2 + 20t(\Delta t)] - 10t^2}{t + \Delta t - t}$$

$$\frac{\Delta s}{\Delta t} = 20t$$

The value of the derivative *ds/dt* in the case plotted in Figure 5 equals 20*t*. Thus, at time $t = 1$ *s*, the instantaneous velocity equals 20 ft/s; at time $t = 2$ *s*, the velocity equals 40 ft/s, and so on.

When the graph of a function is not a straight line, the slope of the plot is different at different points. The slope of a curve at any point is defined as the slope of a line drawn tangent to the curve at that point. Figure 6 shows a line drawn tangent to a curve. A tangent line is a line that touches the curve at only one point. The line *AB* is tangent to the

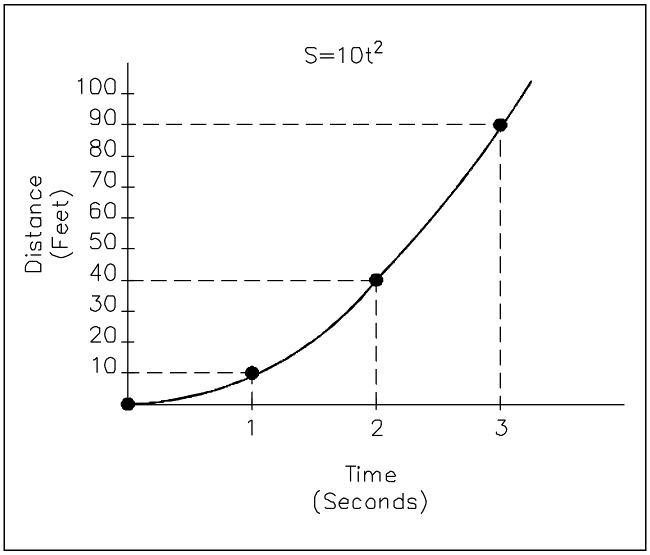

Figure 5 Graph of Distance vs. Time

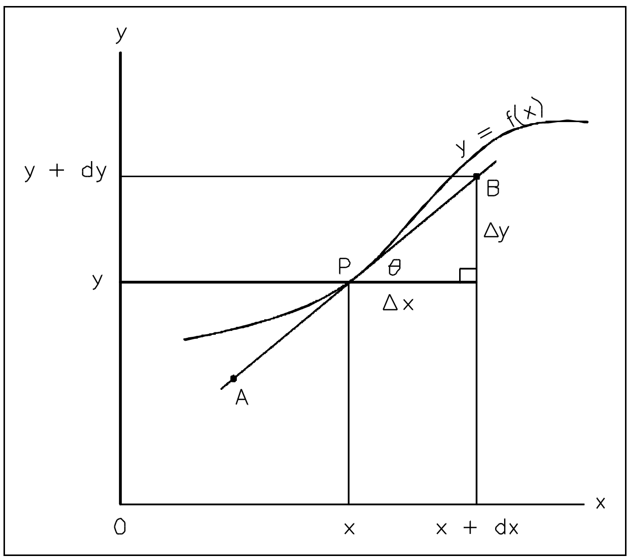

Figure 6 Slope of a Curve

curve $y = f(x)$ at point P.

The tangent line has the slope of the curve dy/dx, where, θ is the angle between the tangent line AB and a line parallel to the x-axis. But, tan θ also equals $\Delta y/\Delta x$ for the tangent line AB, and $\Delta y/\Delta x$ is the slope of the line. Thus, the slope of a curve at any point equals the slope of the line drawn tangent to the curve at that point. This slope, in turn, equals the derivative of y with respect to x, dy/dx, evaluated at the same point.

These applications suggest that a derivative can be visualized as the slope of a graphical plot. A derivative represents the rate of change of one quantity with respect to another. When the relationship between these two quantities is presented in graphical form, this rate of change equals the slope of the resulting plot.

The mathematics of dynamic systems involves many different operations with the derivatives of functions. In practice, derivatives of functions are not determined by plotting the functions and finding the slopes of tangent lines. Although this approach could be used, techniques have been developed that permit derivatives of functions to be determined directly based on the form of the functions. For example, the derivative of the function $f(x) = c$, where c is a constant, is zero. The graph of a constant function is a horizontal line, and the slope of a horizontal line is zero.

$$f(x) = c$$

$$\frac{d[f(x)]}{dx} = 0 \tag{5-6}$$

The derivative of the function $f(x) = ax + c$ (compare to slope m from general form of linear equation, $y = mx + b$), where a and c are constants, is a. The graph of such a function is a straight line having a slope equal to a.

$$f(x) = ax + c$$

$$\frac{d[f(x)]}{dx} = a \tag{5-7}$$

The derivative of the function $f(x) = ax^n$, where a and n are constants, is nax^{n-1}.

$$f(x) = ax^n$$

$$\frac{d[f(x)]}{dx} = nax^{n-1} \tag{5-8}$$

The derivative of the function $f(x) = ae^{bx}$, where a and b are constants and e is the base of natural logarithms, is abe^{bx}.

$$f(x) = ae^{bx}$$

$$\frac{d[f(x)]}{dx} = abe^{bx} \qquad (5\text{-}9)$$

These general techniques for finding the derivatives of functions are important for those who perform detailed mathematical calculations for dynamic systems. For example, the designers of nuclear facility systems need an understanding of these techniques, because these techniques are not encountered in the day-to-day operation of a nuclear facility. As a result, the operators of these facilities should understand what derivatives are in terms of a rate of change and a slope of a graph, but they will not normally be required to find the derivatives of functions.

The notation $d[f(x)]/dx$ is the common way to indicate the derivative of a function. In some applications, the notation $f^{\prime}(x)$ is used. In other applications, the so-called dot notation is used to indicate the derivative of a function with respect to time. For example, the derivative of the amount of heat transferred, Q, with respect to time, *dQ/dt*, is often written as $\dot{Q}$.

It is also of interest to note that many detailed calculations for dynamic systems involve not only one derivative of a function, but several successive derivatives. The second derivative of a function is the derivative of its derivative; the third derivative is the derivative of the second derivative, and so on. For example, velocity is the first derivative of distance traveled with respect to time, $v = ds/dt$; acceleration is the derivative of velocity with respect to time, $a = dv/dt$. Thus, acceleration is the second derivative of distance traveled with respect to time. This is written as d^2s/dt^2. The notation $d^2[f(x)]/dx^2$ is the common way to indicate the second derivative of a function. In some applications, the notation $f^{\prime\prime}(x)$ is used. The notation for third, fourth, and higher order derivatives follows this same format. Dot notation can also be used for higher order derivatives with respect to time. Two dots indicates the second derivative, three dots the third derivative, and so on.

Application of Derivatives to Physical Systems

There are many different problems involving dynamic physical systems that are most readily solved using derivatives. One of the best illustrations of the application of derivatives are problems involving related rates of change. When two quantities are related by some known physical relationship, their rates of change with respect to a third quantity are also related. For example, the area of a circle is related to its radius by the formula $A = \pi r^2$. If for some reason the size of a circle is changing in time, the rate of change of its area, with respect to time for example, is also related to the rate of change of its radius with respect to time. Although these applications involve finding the derivative of function, they illustrate why derivatives are needed to solve certain problems involving dynamic systems.

Example 1:

A stone is dropped into a quiet lake, and waves move in circles outward from the location of the splash at a constant velocity of 0.5 feet per second. Determine the rate at which the area of the circle is increasing when the radius is 4 feet.

Solution:

Using the formula for the area of a circle,

$$A = \pi r^2$$

take the derivative of both sides of this equation with respect to time *t*.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

But, *dr/dt* is the velocity of the circle moving outward which equals 0.5 ft/s and *dA/dt* is the rate at which the area is increasing, which is the quantity to be determined. Set *r* equal to 4 feet, substitute the known values into the equation, and solve for *dA/dt*.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = (2)(3.1416)(4\ \text{ft})(0.5\ \text{ft/s})$$

$$\frac{dA}{dt} = 12.6\ \text{ft}^2/\text{s}$$

Thus, at a radius of 4 feet, the area is increasing at a rate of 12.6 square feet per second.

Example 2:

A ladder 26 feet long is leaning against a wall. The ladder starts to move such that the bottom end moves away from the wall at a constant velocity of 2 feet per second. What is the downward velocity of the top end of the ladder when the bottom end is 10 feet from the wall?

Solution:

Start with the Pythagorean Theorem for a right triangle: $a^2 = c^2 - b^2$

Take the derivative of both sides of this equation with respect to time *t*. The *c*, representing the length of the ladder, is a constant.

$$2a\frac{da}{dt} = -2b\frac{db}{dt}$$

$$a\frac{da}{dt} = -b\frac{db}{dt}$$

But, *db/dt* is the velocity at which the bottom end of the ladder is moving away from the wall, equal to 2 ft/s, and *da/dt* is the downward velocity of the top end of the ladder along the wall, which is the quantity to be determined. Set *b* equal to 10 feet, substitute the known values into the equation, and solve for *a.*

$$a^2 = c^2 - b^2$$

$$a = \sqrt{c^2 - b^2}$$

$$a = \sqrt{(26\ \text{ft})^2 - (10\ \text{ft})^2}$$

$$a = \sqrt{676\ \text{ft}^2 - 100\ \text{ft}^2}$$

$$a = \sqrt{576\ \text{ft}^2}$$

$$a = 24\ \text{ft}$$

$$a\frac{da}{dt} = -b\frac{db}{dt}$$

$$\frac{da}{dt} = -\frac{b}{a}\frac{db}{dt}$$

$$\frac{da}{dt} = -\frac{10\ \text{ft}}{24\ \text{ft}}(2\ \text{ft/s})$$

$$\frac{da}{dt} = -0.833\ \text{ft/s}$$

Thus, when the bottom of the ladder is 10 feet from the wall and moving at 2ft/sec., the top of the ladder is moving downward at 0.833 ft/s. (The negative sign indicates the downward direction.)

Integrals and Summations in Physical Systems

Differentials and derivatives arose in physical systems when small changes in one quantity were considered. For example, the relationship between position and time for a moving object led to the definition of the instantaneous velocity, as the derivative of the distance traveled with respect to time, *ds/dt*. In many physical systems, rates of change are measured directly. Solving problems, when this is the case, involves another aspect of the mathematics of dynamic systems; namely integral and summations.

Figure 7 is a graph of the instantaneous velocity of an object as a function of elapsed time. This is the type of graph which could be generated if the reading of the speedometer of a car were recorded as a function of time.

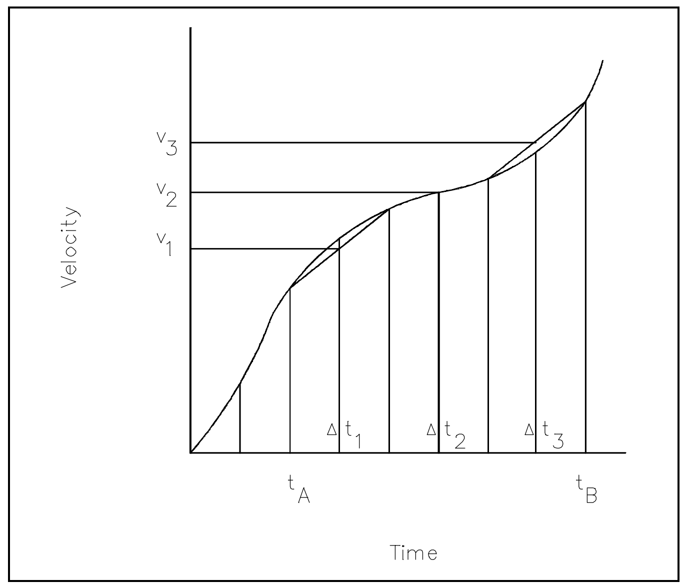

Figure 7 Graph of Velocity vs. Time

At any given instant of time, the velocity of the object can be determined by referring to Figure 7. However, if the distance traveled in a certain interval of time is to be determined, some new techniques must be used. Consider the velocity versus time curve of Figure 7. Let's consider the velocity changes between times t_A and t_B. The first approach is to divide the time interval into three short intervals (Δt_1,Δt_2,Δt_3), and to assume that the velocity is constant during each of these intervals. During time interval Δt_1, the velocity is assumed constant at an average velocity v_1; during the interval Δt_2, the velocity is assumed constant at an average velocity v_2; during time interval Δt_3, the velocity is assumed constant at an average velocity v_3. Then the total distance traveled is approximately the sum of the products of the velocity and the elapsed time over each of the three intervals. Equation 5-10 approximates the distance traveled during the time interval from t_a to t_b and represents the approximate area under the velocity curve during this same time interval.

$$s = v_1 \Delta t_1 + v_2 \Delta t_2 + v_3 \Delta t_3 \tag{5-10}$$

This type of expression is called a summation. A summation indicates the sum of a series of similar quantities. The upper case Greek letter Sigma, Σ, is used to indicate a summation. Generalized subscripts are used to simplify writing summations. For example, the summation given in Equation 5-10 would be written in the following manner:

$$S = \sum_{i=1}^{3} v_i \Delta t_i \tag{5-11}$$

The number below the summation sign indicates the value of *i* in the first term of the summation; the number above the summation sign indicates the value of *i* in the last term of the summation.

The summation that results from dividing the time interval into three smaller intervals, as shown in Figure 7, only approximates the distance traveled. However, if the time interval is divided into incremental intervals, an exact answer can be obtained. When this is done, the distance traveled would be written as a summation with an indefinite number of terms.

$$S = \sum_{i=1}^{\infty} v_i \Delta t_i \tag{5-12}$$

This expression defines an integral. The symbol for an integral is an elongated "s" $\int$. Using an integral, Equation 5-12 would be written in the following manner:

$$S = \int_{t_A}^{t_B} v \, dt \tag{5-13}$$

This expression is read as S equals the integral of $v\ dt$ from $t = t_A$ to $t = t_B$. The numbers below and above the integral sign are the limits of the integral. The limits of an integral indicate the values at which the summation process, indicated by the integral, begins and ends.

As with differentials and derivatives, one of the most important parts of understanding integrals is having a physical interpretation of their meaning. For example, when a relationship is written as an integral, the physical meaning, in terms of a summation, should be readily understood. In the previous example, the distance traveled between t_A and t_B was approximated by equation 5-10. Equation 5-13 represents the exact distance traveled and also represents the exact area under the curve on figure 7 between t_A and t_B.

Examples:

1. Give the physical interpretation of the following equation relating the work, W, done when a force, F, moves a body from position x_1 to x_2.

$$W = \int_{x_1}^{x_2} F\ dx$$

The physical meaning of this equation can be stated in terms of a summation. The total amount of work done equals the integral of $F\ dx$ from $x = x_1$ to $x = x_2$. This can be visualized as taking the product of the instantaneous force, F, and the incremental change in position dx at each point between x_1 and x_2, and summing all of these products.

2. Give the physical interpretation of the following equation relating the amount of radioactive material present as a function of the elapsed time, t, and the decay constant, λ.

$$\int_{N_0}^{N_1} \frac{dN}{N} = -\lambda t$$

The physical meaning of this equation can be stated in terms of a summation. The negative of the product of the decay constant, λ, and the elapsed time, t, equals the integral of dN/N from $N = N_0$ to $n = n_1$. This integral can be visualized as taking the quotient of the incremental change in N, divided by the value of N at each point between N_0 and N_1, and summing all of these quotients.

Graphical Understanding of Integral

As with derivatives, when a functional relationship is presented in graphical form, an important understanding of the meaning of integral can be developed.

Figure 8 is a plot of the instantaneous velocity, v, of an object as a function of elapsed time, t. The functional relationship shown is given by the following equation:

$$v = 6t \tag{5-14}$$

The distance traveled, s, between times t_A and t_B equals the integral of the velocity, v, with respect to time between the limits t_A and t_B.

$$s = \int_{t_A}^{t_B} v\ dt \tag{5-15}$$

The value of this integral can be determined for the case plotted in Figure 8 by noting that the velocity is increasing linearly. Thus, the average velocity for the time interval between t_A and t_B is the arithmetic average of the velocity at t_A and the velocity at t_B. At time t_A, $v = 6t_A$; at time t_B, $v = 6t_B$. Thus, the average velocity for the time interval between t_A and t_B is $\dfrac{6t_A + 6t_B}{2}$ which equals $3(t_A + t_B)$. Using this average velocity, the total distance traveled in the time interval between t_A and t_B is the product of the elapsed time $t_B - t_A$ and the average velocity $3(t_A + t_B)$.

$$s = v_{av}\Delta t$$

$$s = 3(t_A + t_B)(t_B - t_A) \qquad (5\text{-}16)$$

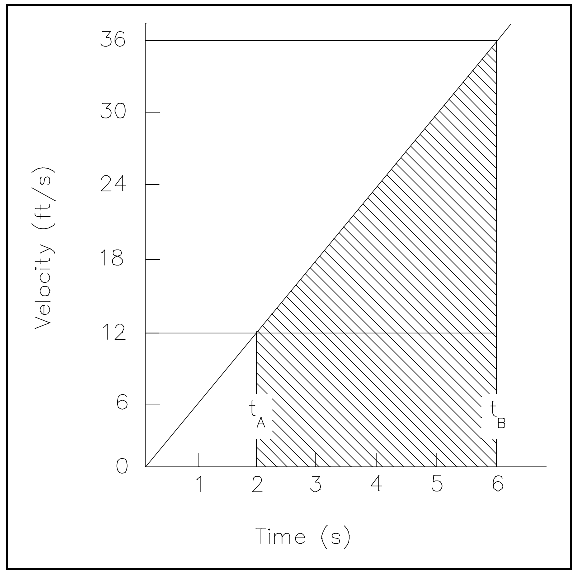

Figure 8 Graph of Velocity vs. Time

Equation 5-16 is also the value of the integral of the velocity, v, with respect to time, t, between the limits t_A -t_B for the case plotted in Figure 8.

$$\int_{t_A}^{t_B} v\,dt = 3(t_A + t_B)(t_B - t_A)$$

The cross-hatched area in Figure 8 is the area under the velocity curve between $t = t_A$ and $t = t_B$. The value of this area can be computed by adding the area of the rectangle whose sides are $t_B - t_A$ and the velocity at t_A, which equals $6t_A - t_B$, and the area of the triangle whose base is $t_B - t_A$ and whose height is the difference between the velocity at t_B and the velocity at t_A, which equals $6t_B - t_A$.

$$\text{Area} = [(t_B - t_A)(6t_A)] + \left[\frac{1}{2}(t_B - t_A)(6t_b - 6t_A)\right]$$

$$\text{Area} = 6t_A t_B - 6t_A^2 + 3t_B^2 - 6t_A t_B + 3t_A^2$$

$$\text{Area} = 3t_B^2 - 3t_A^2$$

$$\text{Area} = 3(t_B + t_A)(t_B - t_A)$$

This is exactly equal to the value of the integral of the velocity with respect to time between the limits t_A and t_B. Since the distance traveled equals the integral of the velocity with respect to time, $\int v\,dt$, and since this integral equals the area under the curve of velocity versus time, the distance traveled can be visualized as the area under the curve of velocity versus time.

For the case shown in Figure 8, the velocity is increasing at a constant rate. When the plot of a function is not a straight line, the area under the curve is more difficult to determine. However, it can be shown that the integral of a function equals the area between the x-axis and the graphical plot of the function.

$$\int_{X_1}^{X_2} f(x)dx = \text{Area between } f(x) \text{ and x-axis from } x_1 \text{ to } x_2$$

The mathematics of dynamic systems involves many different operations with the integral of functions. As with derivatives, in practice, the integral of functions are not determined by plotting the functions and measuring the area under the curves. Although this approach could be used, techniques have been developed which permit integral of functions to be determined directly based on the form of the functions. Actually, the technique for taking an integral is the reverse of taking a derivative. For example, the derivative of the function $f(x) = ax + c$, where a and c are constants, is a. The integral of the function $f(x) = a$, where a is a constant, is $ax + c$, where a and c are constants.

$$f(x) = a$$

$$\int f(x)dx = ax + c \tag{5-17}$$

The integral of the function $f(x) = ax^n$, where a and n are constants, is $\frac{a}{n+1}x^{n+1} + c$, where c is another constant.

$$f(x) = ax^n$$

$$\int f(x)dx = \frac{a}{n+1}x^{n+1} + c \tag{5-18}$$

The integral of the function $f(x) = ae^{bx}$, where a and b are constants and e is the base of natural logarithms, is $\frac{ae^{bx}}{b} + c$, where c is another constant.

$$f(x) = ae^{bx}$$

$$\int f(x)dx = \frac{a}{b}e^{bx} + c \qquad (5\text{-}19)$$

As with the techniques for finding the derivatives of functions, these general techniques for finding the integral of functions are primarily important only to those who perform detailed mathematical calculations for dynamic systems. These techniques are not encountered in the day-to-day operation of a nuclear facility. However, it is worthwhile to understand that taking an integral is the reverse of taking a derivative. It is important to understand what integral and derivatives are in terms of summations and areas under graphical plot, rates of change, and slopes of graphical plots.

Summary

The important information covered in this chapter is summarized below.

Derivatives and Differentials Summary

- The derivative of a function is defined as the rate of change of one quantity with respect to another, which is the slope of the function.
- The integral of a function is defined as the area under the curve.